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Propagation of shear elastic waves in composites with a random set of spherical inclusions (effective field approach)

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Abstract

The work is dedicated to the problem of plane monochromatic shear wave propagation through elastic matrix composite materials with a homogeneous random set of spherical inclusions. The effective field method (EFM) and quasi-crystalline approximation are used for the calculation of phase velocity and attenuation factor of the mean wave field propagating through the composite. The version of the method developed in the work allows us to obtain the dispersion equation for the wave vector of the mean wave field that serves for all frequencies of the incident field, properties and volume concentrations of the inclusions. The long- and short-wave asymptotic solutions of the dispersion equation are found in closed analytical forms. Numerical solutions of this equation are constructed in a wide region of frequencies that covers the long-, middle- and short-wave regions of the propagating waves. The phase velocities and attenuation factors of the mean wave field in the composites are analyzed for various elastic properties, density and volume concentrations of the inclusions. Comparisons of the predictions of the method with some numerical computation of the effective parameters of matrix composites are presented; possible errors in predictions of the velocities and attenuation factors of the mean wave field in the composites are indicated and discussed.

Keywords: Elastic composites; Homogenization problem; Effective field method; Mean wave field; Phase velocity; Attenuation factor; Dispersion

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1. Introduction

The problem of wave propagation through heterogeneous media has a number of important applications such as non-destructive evaluation of the microstructures of composite materials and prediction of the dynamic properties of composites. In this work the problem of monochromatic wave propagation through the medium with a set of isolated spherical inclusions is considered. If the set of inclusions is random, the exact solution of this problem cannot be found, and only various approximations are available. A version of the effective field method (EFM) is developed in this work for the construction of such an approximate solution. This method has a long history and was successfully used in the nuclear physics, in the theory of phase transitions for the description of various physical phenomena in the ensembles of interacting particles. An important area of its application is the problem of wave propagation through the medium with isolated inclusions (scatterers). The main aim of the theory in this problem is prediction of the phase velocity and attenuation factor of the mean (coherent) wave field propagating through the inhomogeneous medium.

The EFM is based on some hypotheses about the structure of a local external field that acts on every particle (inclusion) in the composite medium. As a rule the area of the application of these hypotheses cannot be strictly indicated, and only comparison with experimental data or numerical solutions allow us to point out the borders of this area. In order to understand the character of possible errors of the method it is important to analyze its predictions in a wide region of frequencies of the incident field and for various elastic properties, densities and volume concentrations of inclusions.

Application of the EFM to the solution of the problem of wave propagation through inhomogeneous medium starts with the famous work of Rayleigh (1892). The hypotheses of the EFM were formulated explicitly in the classical works of Foldy (1945), Lax (1951) and Lax (1952), where the method was applied to the problem of scalar wave propagation through the medium with point scatterers. It was assumed in these works that the local (effective) field that acts on every scatterer in the medium is a plane wave, and it is the same for all the scatterers. This wave was supposed to be coincided with mean wave field (Foldy) or proportional to the mean field (Lax). This hypothesis was called the quasi-crystalline approximation, and in many works this name is associated with the method itself. This hypothesis reduces the solution of the many particle problem (interaction between many particles) to a one particle problem (diffraction of effective external field on one particle). Another version of the EFM for the case of scalar waves was developed in the works of Waterman and Truel (1961) and Ficioris and Waterman (1964), where the effective field was assumed to be a combination of the forward and backward plane waves with the wave numbers of the background medium (matrix).

Application of the quasi-crystalline approximation to the problem of wave propagation through elastic media with isolated inclusions encounters two main difficulties. Firstly, the one particle problem in this case is diffraction of a plane monochromatic wave on an inclusion of finite sizes, and the exact solution of this problem for an arbitrary frequency of the incident field exists only for a spherical inclusion. In the long-wave region, where the one particle problem is quasistatic (lengths of the propagating waves are more than the characteristic size of inclusions), the exact solution may be found for an arbitrary ellipsoidal homogeneous inclusion and its limit forms. In the case of inclusions of non-canonical shapes only numerical solutions of the one particle problem are available.

The second difficulty in the application of the method is the procedure of averaging the detailed wave field in the composite over the ensemble realizations of the random field of inclusions. In a number of works where the EFM was applied to the problem of elastic wave propagation in particulate composites, the long-wave region and spherical or cylindrical inclusions were considered (see Bose and Mal, 1973, 1974; Datta, 1977; Datta et al., 1988, and others). The technique that was used in these works was the expansion of the fields scattered on inclusions over eigenfunctions of the one particle diffraction problem. In the works of Varadan et al. (1978) and Varadan and Varadan (1985) this technique was used for the analysis of wave propagation in the middle wave length region. The main drawback of this technique is the complexity of the procedure of ensemble averaging. It was shown in the works of Twersky (1975, 1978) and Willis (1980) that the technique of integral equations is a more efficient tool for the realization of the averaging procedure.

In this work we consider propagation of monochromatic shear waves in the medium with spherical inclusions and develop the mathematical formalism of the EFM that serves for all frequencies of the incident field, elastic properties, density and volume concentrations of inhomogeneities. Using this formalism we obtain the dispersion equation for the wave number of the mean wave field in the composite, and the real and imaginary parts of this number give us the phase velocity and attenuation factor of the mean wave field. Explicit asymptotic solutions of the dispersion equation are found in the long- and short-wave regions; numerical solutions of this equation are constructed in a wide region of frequencies of the incident field and for various properties and volume concentrations of inclusions. The method is based on the hypotheses that are close to the Lax version of the EFM. The structure of the paper is as follows.

In Section 2, we consider the integral equations of the problem of monochromatic wave propagation through matrix composite materials. In Section 3, the main hypotheses of the EFM are formulated, and the general scheme of the method is developed. The dispersion equation for the wave vector of the mean wave field in the composite that serves for all frequencies of the incident field, properties and volume concentrations of inclusions is obtained in this section. The coefficients in the dispersion equation are expressed via the solution of the one-particle problem (the problem of diffraction of the effective external field on an isolated spherical particle). The solution of this problem and the final form of the dispersion equation are presented in Section 4. In Section 5, we obtain the long-wave asymptotic solution of the dispersion equation in an explicit analytical form. The comparisons of the predictions of the method with numerical calculations of the effective elastic constants of the composites are presented in this section. In Section 6, the short-wave asymptotic of the solution of dispersion equation is obtained and discussed. Section 7 is dedicated to the numerical solution of the dispersion equation in a wide region of frequencies that covers long-, middle- and short-wave regions. We consider two types of inclusions that are much harder and heavier than the matrix and much lighter and softer than the matrix. It is shown that the dispersion equation has several branches of its solutions. The main branch may be interpreted as an acoustic (quasiacoustic) one, and some other branches (e.g., quasioptical) may be also indicated. In the conclusion (Section 8) the area of the application of the method is discussed.

2. Integral equations of the diffraction problem

Let us consider an infinite homogeneous medium (matrix) with elastic moduli C^0 and mass density ρ_0 containing a homogeneous random set of inclusions with elastic moduli tensor C and mass density ρ. The inclusions occupy region V, and $V(x)$ is the characteristic function of this region ($V(x) = 1$ if $x \in V$ and $V(x) = 0$ if $x \notin V$). Here, $x(x_1, x_2, x_3)$ is a point of the medium with Cartesian coordinates x_1, x_2, x_3. We study a monochromatic elastic wave of frequency ω that propagates in such a medium. If the dependence of time t is defined by the factor $\exp(-\mathrm{i}\omega t)$, the displacement field u_i in the medium has the form $u_i(x,t) = u_i(x)\exp(-\mathrm{i}\omega t)$, and amplitude $u_i(x)$ of this field satisfies the following integral equation (see, e.g., Willis, 1980)

$$\begin{aligned} &u_i(x) = u_i^0(x) + \int \partial_j G_{ik}(x-x')C^1_{kjmn}(x')\varepsilon_{mn}(x')\,\mathrm{d}x' + \omega^2 \int G_{ik}(x-x')\rho_1(x')u_k(x')\,\mathrm{d}x', \partial_i = \frac{\partial}{\partial x_i}, \\ &C^1(x) = C^1 V(x), \quad C^1 = C - C^0, \quad \rho_1(x) = \rho_1 V(x), \quad \rho_1 = \rho - \rho_0. \end{aligned} \tag{2.1}$$

Here, $u_i^0(x)$ is an incident field that would have existed in the matrix without inclusions under prescribed conditions at infinity, $\varepsilon_{ij} = \partial_{(i}u_{j)}$ is the strain tensor, and $G_{ik}(x)$ is the Green function of the operator $\partial_j C^0_{ijkl}\partial_l + \rho_0\omega^2\delta_{ik}$. For the isotropic medium with λ_0, μ_0 as Lamé parameters, tensor $G_{ik}(x)$ takes the form

$$G_{ik}(x) = \frac{1}{4\pi\rho_0\omega^2}\left[\delta_{ik}\beta_0^2\frac{e^{i\beta_0 r}}{r} - \partial_i\partial_k\left(\frac{e^{i\alpha_0 r}}{r} - \frac{e^{i\beta_0 r}}{r}\right)\right],$$
$$\alpha_0^2 = \frac{\omega^2\rho_0}{\lambda_0 + 2\mu_0}, \quad \beta_0^2 = \frac{\omega^2\rho_0}{\mu_0}, \tag{2.2}$$

where δ_{ik} is Kroneker's symbol.

It follows from Eq. (2.1) that the amplitude $\varepsilon_{ij}(x)$ of the strain tensor in the composite medium satisfies the equation

$$\varepsilon_{ij}(x) = \varepsilon_{ij}^0 + \int P_{ijkl}(x-x')C_{klmn}^1(x')\varepsilon_{mn}(x')\,dx' + \omega^2\int \partial_{(j}G_{i)k}(x-x')\rho_1(x')u_k(x')\,dx',$$
$$P_{ijkl}(x) = \frac{\partial^2 G_{ik}}{\partial x_j \partial x_l}\Big|_{(ij)(kl)}. \tag{2.3}$$

Here, parentheses in indices means symmetrization: $T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$.

If the incident field is a plane monochromatic wave, displacement vector $u_i^0(x)$ and strain tensor $\varepsilon_{ij}^0(x)$ take the forms

$$u_i^0(x) = U_i^0 e^{i\mathbf{q}_0\cdot\mathbf{x}}, \quad \varepsilon_{ij}^0(x) = iq^0{}_{(i}U_{j)}^0 e^{i\mathbf{q}_0\cdot\mathbf{x}},$$
$$\mathbf{q}_0 = q_0\mathbf{n}_0, \quad \mathbf{q}_0\cdot\mathbf{x} = q_0 n_i^0 x_i, \tag{2.4}$$

where q_0 is the wave number of the incident wave in the matrix, $\mathbf{n}^0$ is the wave normal and U_i^0 is the polarization vector ($q_0 = \alpha_0$ for longitudinal waves and $q_0 = \beta_0$ for shear waves).

Note that the functions $C^1(x)$ and $\rho_1(x)$ in the right-hand side of Eqs. (2.1) and (2.3) cut the functions $u_i(x)$ and $\varepsilon_{ij}(x)$ on the region V occupied by the inclusions. Thus, the main unknowns of the problem are the values of these fields inside the inclusions. The fields in the matrix may be reconstructed from Eqs. (2.1) and (2.3) if the fields $u_i(x)$, $\varepsilon_{ij}(x)$ inside the inclusions are known.

3. General scheme of the effective field method

Let us consider a typical realization of a homogeneous random set of inclusions in the background medium (matrix). Every inclusion in the composite may be considered as an isolated one in the original matrix by the action of local external displacement $u_i^*(x)$ and strain $\varepsilon_{ij}^*(x)$ fields. The fields $u_i^*(x)$ and $\varepsilon_{ij}^*(x)$ do not coincide with the incident fields $u_i^0(x)$ and $\varepsilon_{ij}^0(x)$ applied to the medium; $u_i^*(x)$ and $\varepsilon_{ij}^*(x)$ consist of the incident fields and the fields scattered on the surrounding inclusions. If an inclusion occupies region v, the fields inside this inclusion satisfies the integral equations that are similar to Eqs. (2.1) and (2.3) ($x \in v$)

$$u_i(x) = u_i^*(x) + \int_v \partial_j G_{ik}(x-x')C_{kjmn}^1\varepsilon_{mn}(x')\,dx' + \rho_1\omega^2\int_v G_{ik}(x-x')u_k(x')\,dx', \tag{3.1}$$

$$\varepsilon_{ij}(x) = \varepsilon_{ij}^*(x) + \int_v P_{ijkl}(x-x')C_{klmn}^1\varepsilon_{mn}(x')\,dx' + \rho_1\omega^2\int_v \partial_{(j}G_{i)k}(x-x')u_k(x')\,dx'. \tag{3.2}$$

3.1. Integral equations for the effective fields

Let Eqs. (3.1) and (3.2) may be solved for arbitrary external fields $u_i^*(x)$, $\varepsilon_{ij}^*(x)$, and the fields $u_i^{(k)}(x)$, $\varepsilon_{ij}^{(k)}(x)$ inside the inclusion centered at point x^k may be presented in the form

$$u_i^{(k)}(x) = \lambda_{ik}^{(k)}u_k^*(x), \quad \varepsilon_{ij}^{(k)}(x) = \Lambda_{ijkl}^{(k)}\varepsilon_{kl}^*(x), \tag{3.3}$$

where $\lambda_{ik}^{(k)}$ and $\Lambda_{ijkl}^{(k)}$ are some linear operators of the solution of the diffraction problem for one inclusion (one particle problem). It follows from Eqs. (2.1) and (2.3) that the fields $u_i(x)$ and $\varepsilon_{ij}(x)$ in the medium are expressed via the fields $u_i^*(x)$ and $\varepsilon_{kl}^*(x)$ in the form

$$u_i(x) = u_i^0(x) + \int \left[\partial_j G_{ik}(x-x')C_{kjmn}^1 \Lambda_{mnrs}\varepsilon_{rs}^*(x') + \rho_1\omega^2 G_{ik}(x-x')\lambda_{kl}u_l^*(x')\right]V(x')\,\mathrm{d}x', \tag{3.4}$$

$$\varepsilon_{ij}(x) = \varepsilon_{ij}^0(x) + \int \left[P_{ijkl}(x-x')C_{klmn}^1 \Lambda_{mnrs}\varepsilon_{rs}^*(x') + \rho_1\omega^2 \partial_{(i}G_{j)k}(x-x')\lambda_{kl}u_l^*(x')\right]V(x')\,\mathrm{d}x'. \tag{3.5}$$

Here, functions $\Lambda_{mnrs}\varepsilon_{rs}^*(x)$ and $\lambda_{kl}u_l^*(x)$ coincide with $\Lambda_{mnrs}^{(k)}\varepsilon_{rs}^*(x)$ and $\lambda_{kl}^{(k)}u_l^*(x)$ inside the k-th inclusion ($k = 1, 2, 3, \ldots$). Linear operators $\Lambda_{mnrs}^{(k)}$ and $\lambda_{kl}^{(k)}$ may be presented in the forms of some integral operators

$$\Lambda_{mnrs}^{(k)}\varepsilon_{rs}^*(x) = \int_{v_k} \Lambda_{mnrs}^{(k)}(x,x')\varepsilon_{rs}^*(x')\,\mathrm{d}x', \tag{3.6}$$

$$\lambda_{kl}^{(k)}u_l^*(x) = \int_{v_k} \lambda_{kl}^{(k)}(x,x')u_l^*(x')\,\mathrm{d}x', \tag{3.7}$$

where $\Lambda_{mnrs}^{(k)}(x,x')$ and $\lambda_{kl}^{(k)}(x,x')$ are generalized functions known from the solution of the one particle problem. Points x and x' belong to the same domain v_k because the fields inside k-th inclusion depend only on the values of the local external fields in the region v_k.

The equations for the local external fields $u_i^*(x)$ and $\varepsilon_{ij}^*(x)$ that act on an arbitrary (k-th) inclusion follow from their definitions as a sum of the incident fields and the fields scattered on surrounding inclusions and take the forms

$$u_i^*(x) = u_i^0(x) + \int \left[\partial_j G_{ik}(x-x')C_{kjmn}^1 \Lambda_{mnrs}\varepsilon_{rs}^*(x') + \rho_1\omega^2 G_{ik}(x-x')\lambda_{kl}u_l^*(x')\right]V(x;x')\,\mathrm{d}x', \tag{3.8}$$

$$\varepsilon_{ij}^*(x) = \varepsilon_{ij}^0(x) + \int \left[P_{ijkl}(x-x')C_{klmn}^1 \Lambda_{mnrs}\varepsilon_{rs}^*(x') + \rho_1\omega^2 \partial_{(j}G_{i)k}(x-x')\lambda_{kl}u_l^*(x')\right]V(x;x')\,\mathrm{d}x'. \tag{3.9}$$

Here, $V(x;x')$ is the characteristic function (with argument x') of region V_x defined by the equation

$$V_x = \bigcup_{i \neq j} v_i \quad \text{if } x \in v_j. \tag{3.10}$$

As it follows from Eq. (3.10) function $V(x;x')$ is equal to zero if points x and x' are inside the same inclusion. Thus, the integral terms in Eqs. (3.8) and (3.9) are the sums of the fields scattered on all the inclusions except the one that occupies region v_j if $x \in v_j$.

3.2. The main hypotheses of the EFM and average procedure

It is seen from Eqs. (3.4), (3.5), (3.8) and (3.9) that the local external fields $u_l^*(x)$ and $\varepsilon_{rs}^*(x)$ may be considered as the main unknowns of the problem. For random sets of inhomogeneities $u_l^*(x)$ and $\varepsilon_{rs}^*(x)$ are random functions. The main hypotheses of the EFM concern the structure of the fields $u_l^*(x)$ and $\varepsilon_{rs}^*(x)$. Let us introduce the first of hypothesis H1.

H1. The fields $u_l^*(x)$ and $\varepsilon_{rs}^*(x)$ are plane waves in the vicinity of every inclusion.

This hypothesis allows us to construct operators Λ and λ in Eqs. (3.6) and (3.7) from the solution of the problem of diffraction of a plane monochromatic wave on an isolated inclusion (see Section 4).

Let us find the mean wave field $\langle u_i(x)\rangle$ in the composite medium. After averaging Eq. (3.4) over the ensemble realizations of the random sets of inclusions, we obtain

$$\langle u_i(x)\rangle = u_i^0(x) + \int \left[\partial_j G_{ik}(x-x')C^1_{kjmn}\left\langle \Lambda_{mnrs}\varepsilon^*_{rs}(x')V(x')\right\rangle + \rho_1\omega^2 G_{ik}(x-x')\left\langle \lambda_{kl}u_l^*(x')V(x')\right\rangle\right] \mathrm{d}x'. \tag{3.11}$$

A similar equation may be written for the mean value of the strain field $\langle \varepsilon_{ij}(x)\rangle$ after averaging of Eq. (3.5).

For the next step we have to introduce the second hypothesis of the EFM.

H2. Random local external fields $u_i^*(x)$ *and* $\varepsilon^*_{ij}(x)$ acting on an inclusion v are statistically independent on the elastic properties, density and radius of this inclusion.

Thus, it is assume in this hypothesis that the local external field acting on an arbitrary inclusion in the composite depends mainly on global statistical characteristics of the random set of inhomogeneities, and it is not sensitive to the properties of individual inclusions.

Hypothesis H2 allows us to write the mean values of the functions $\Lambda_{ijkl}\varepsilon^*_{kl}(x)V(x)$ and $\lambda_{ik}u_k^*(x)V(x)$ in the right-hand side of Eq. (3.11) in the forms

$$\left\langle \Lambda_{ijkl}\varepsilon^*_{kl}(x)V(x)\right\rangle = \left\langle V(x)\Lambda_{ijkl}\right\rangle\left\langle \varepsilon^*_{kl}(x)|x\right\rangle, \tag{3.12}$$

$$\left\langle \lambda_{ik}u_k^*(x)V(x)\right\rangle = \langle V(x)\lambda_{ik}\rangle\left\langle u_k^*(x)|x\right\rangle. \tag{3.13}$$

Here, $\langle\cdot|x\rangle$ is averaging under the condition that point x belongs to the region V occupied by the inclusions. It follows from Eqs. (3.11)–(3.13) that the mean wave fields $\langle u_i(x)\rangle$ takes the form

$$\langle u_i(x)\rangle = u_i^0(x) + p\int \left[\partial_j G_{ik}(x-x')C_{kjmn}\Lambda^0_{mnrs}\hat{\varepsilon}^*_{rs}(x') + \rho_1\omega^2 G_{ik}(x-x')\lambda^0_{kl}\hat{u}_l^*(x')\right]\mathrm{d}x', \tag{3.14}$$

$$\hat{\varepsilon}^*_{rs}(x) = \left\langle \varepsilon^*_{kl}(x)|x\right\rangle, \quad \hat{u}_l^*(x) = \left\langle u_k^*(x)x\right\rangle. \tag{3.15}$$

where p is the volume concentration of inclusions. The fields $\hat{u}_l^*(x)$ and $\hat{\varepsilon}^*_{rs}(x)$ are the mean external fields that acts on inclusions in the composite medium. Farther, they will be called effective external fields. Operators Λ^0_{ijkl} and λ^0_{ij} in Eq. (3.14) are defined by the equations

$$\left\langle \Lambda_{ijkl}\varepsilon^*_{kl}(x)V(x)\right\rangle = \int \left\langle V(x')\Lambda_{ijkl}(x,x')\right\rangle\hat{\varepsilon}^*_{kl}(x')\,\mathrm{d}x' = p\int \Lambda^0_{ijkl}(x-x')\hat{\varepsilon}^*_{kl}(x')\,\mathrm{d}x' = p\Lambda^0_{ijkl}\hat{\varepsilon}^*_{kl}(x), \tag{3.16}$$

$$\left\langle \lambda_{ik}u_k^*(x)V(x)\right\rangle = \int \langle V(x')\lambda_{ik}(x,x')\rangle\hat{u}_k^*(x')\,\mathrm{d}x' = p\int \lambda^0_{ik}(x-x')\hat{u}_k^*(x')\,\mathrm{d}x' = p\lambda^0_{ik}\hat{u}_k^*(x). \tag{3.17}$$

The kernels $\Lambda^0_{ijkl}(x-x') = \langle \Lambda_{ijkl}(x,x')|x\rangle$ and $\lambda^0_{ij}(x-x') = \langle \lambda_{ij}(x,x')|x\rangle$ of the operators Λ^0 and λ^0 depend on the difference $x-x'$ for a homogeneous random set of inclusions.

In order to find the means $\hat{u}_i^*(x)$ and $\hat{\varepsilon}^*_{ij}(x)$ in Eqs. (3.14)–(3.17) let us average Eqs. (3.8) and (3.9) over ensemble realizations of the random field of inclusions by the condition that $x\in V$

$$\hat{u}_i^*(x) = u_i^0(x) + \int \left[\partial_j\partial_{(j}G_{i)k}(x-x')C^1_{kjmn}\langle \Lambda_{mnrs}\varepsilon^*_{rs}(x')V(x;x')|x\rangle + \rho_1\omega^2 G_{ik}\langle \lambda_{kl}u_l^*(x')V(x,x')|x\rangle\right]\mathrm{d}x, \tag{3.18}$$

$$\hat{\varepsilon}^*_{ij}(x) = \varepsilon^0_{ij}(x) + \int \left[P_{ijkl}(x-x')C^1_{klmn}\langle \Lambda_{mnrs}\varepsilon^*_{rs}(x')V(x;x')|x\rangle + \rho_1\omega^2\partial_{(j}G_{i)k}\langle \lambda_{kl}u_l^*(x')V(x,x')|x\rangle\right]\mathrm{d}x'. \tag{3.19}$$

Using hypothesis H2 the means in the right-hand sides of these equations may be presented in the forms

$$\langle \Lambda_{mnrs}\varepsilon^*_{rs}(x')V(x;x')|x\rangle = p\langle \Lambda_{mnrs}\varepsilon^*_{rs}(x')|x',x\rangle \Psi(x,x'), \tag{3.20}$$

$$\langle \lambda_{kl}u^*_l(x')V(x,x')|x\rangle = p\langle \lambda_{kl}u^*_l(x')|x',x\rangle \Psi(x,x'), \tag{3.21}$$

$$\Psi(x,x') = \frac{1}{p}\langle V(x;x')|x\rangle. \tag{3.22}$$

Here, function $\Psi(x,x')$ depends only on geometrical properties of the random set of inclusions. If this set is homogeneous and isotropic, $\Psi(x,x')$ is a function of only $|x-x'|$: $\Psi(x,x') = \Psi(|x-x'|)$. The properties of this function follows from Eqs. (3.10) and (3.22): $\Psi(x)$ is a continuous function and

$$\Psi(0) = 0, \quad \Psi(\infty) = 1. \tag{3.23}$$

As it is seen from Eqs. (3.18)–(3.21) the conditional means $\hat{u}^*_i(x) = \langle u^*_i(x)|x\rangle$ and $\hat{\varepsilon}^*_{ij}(x) = \langle \varepsilon^*_{ij}(x)|x\rangle$ are expressed via more complex conditional means $\langle \lambda_{ik}u^*_k(x')|x',x\rangle$ and $\langle \Lambda_{ijkl}\varepsilon^*_{kl}(x')|x',x\rangle$ (the averaging under the condition that points x and x' belong to V). These two-points conditional means can be expressed via three-point similar means using the same Eqs. (3.8) and (3.9), etc. (If we average Eqs. (3.8) and (3.9) by the conditions $x,x' \in V$, in the right-hand sides of these equations appear the means of the functions $u^*_i(x'')$ and $\varepsilon^*_{ij}(x'')$ under the condition that $x'',x',x \in V$). As a result we go to an infinite chain of equations that connects all the multipoint conditional means of the effective fields $u^*_i(x)$ and $\varepsilon^*_{ij}(x)$. In order to obtain a closed system of equations for the means $\langle u^*_i(x)|x\rangle$ and $\langle \varepsilon^*_{ij}(x)|x\rangle$ one has to accept an additional hypothesis H3 concerned the properties of the conditional means. The simplest one is called the quasi-crystalline approximation, and according to this hypothesis we accept

$$\langle \Lambda_{ijkl}\varepsilon^*_{kl}(x')|x',x\rangle = \langle \Lambda_{ijkl}\varepsilon^*_{kl}(x')|x'\rangle = \Lambda^0_{ijkl}\hat{\varepsilon}^*_{kl}(x'), \tag{3.24}$$

$$\langle \lambda_{ik}u^*_k(x')|x',x\rangle = \langle \lambda_{ik}u^*_{kl}(x')|x'\rangle = \lambda^0_{ik}\hat{u}^*_k(x'). \tag{3.25}$$

Here, Λ^0_{ijkl} and λ^0_{ik} are some non-random operators that will be constructed below from the solution of the one particle problem. For the solution of the problem of scalar wave propagation through the medium with point scatterers a similar hypothesis was formulated by Lax (1951, 1952).

Thus, hypothesis H3 may be formulated as follows.

H3. The means of the wave fields $u_i(x') = \lambda_{ik}u^*_k(x')$ and $\varepsilon_{ij}(x') = \Lambda_{ijkl}\varepsilon^*_{kl}(x')$ under the condition that points x' and x are inside different inclusions coincide with the same means by the condition that only point x' is inside of an inclusion ($x' \in V$).

This assumption closes the chain of the equations for many-point conditional means of the effective fields at the first step. Eqs. (3.18) and (3.19) together with Eqs. (3.20)–(3.22), (3.24) and (3.25) give us the following system of integral equations for the fields $\hat{\varepsilon}^*_{kl}(x)$ and $\hat{u}^*_k(x)$

$$\hat{u}^*_i(x) = u^0_i(x) + p\int \left[\partial_j G_{ik}(x-x')C^1_{kjmn}\Lambda^0_{mnrs}\hat{\varepsilon}^*_{rs}(x') + \rho_1\omega^2 G_{ik}(x-x')\lambda^0_{kl}\hat{u}^*_l(x')\right]\Psi(x-x')\,\mathrm{d}x', \tag{3.26}$$

$$\hat{\varepsilon}^*_{ij}(x) = \varepsilon^0_{ij}(x) + p\int \left[P_{ijkl}(x-x')C^1_{klmn}\Lambda^0_{mnrs}\hat{\varepsilon}^*_{rs}(x') + \rho_1\omega^2\partial_{(j}G_{i)k}(x-x')\lambda^0_{kl}\hat{u}^*_l(x')\right]\Psi(x-x')\,\mathrm{d}x'. \tag{3.27}$$

After excluding the incident fields $u^0_i(x)$ and $\varepsilon^0_{ij}(x)$ from these equation and Eq. (3.11) and similar equation for $\langle \varepsilon_{ij}\rangle$ we obtain

$$\hat{u}^*_i(x) = \langle u_i(x)\rangle - p\int \left[\partial_j G_{ik}(x-x')C^1_{kjmn}(\Lambda^0_{mnrs}\hat{\varepsilon}^*_{rs})(x') + \rho_1\omega^2 G_{ik}(x-x')(\lambda^0_{kl}\hat{u}^*_l)(x')\right]\Phi(x-x')\,\mathrm{d}x', \tag{3.28}$$

$$\hat{\varepsilon}^*_{ij}(x) = \langle \varepsilon_{ij}(x)\rangle - p\int \left[P_{ijkl}(x-x')C^1_{klmn}(\Lambda^0_{mnrs}\hat{\varepsilon}^*_{rs})(x') + \rho_1\omega^2\partial_{(j}G_{i)k}(x-x')(\lambda^0_{kl}\hat{u}^*_l)(x')\right]\Phi(x-x')\,\mathrm{d}x', \tag{3.29}$$

$$\Phi(x) = 1 - \Psi(x). \tag{3.30}$$

Note that function $\Phi(x)$ is equal to zero outside a finite vicinity of the origin ($x = 0$). The size of this vicinity has the order of the correlation radius of the random set of inclusions.

3.3. The equations for the effective fields in the case of shear wave propagation

In what follows we will consider the case of shear incident wave $\mathbf{u}_0(x) = U_0\mathbf{m}\exp(\mathrm{i}\beta_0\mathbf{n}\cdot\mathbf{x})$, where $\mathbf{n}$ is wave normal, β_0 is the wave number of the waves in the matrix, U_0 is the amplitude of the incident field, and vectors $\mathbf{m}$ and $\mathbf{n}$ are orthogonal. For a homogeneous and isotropic random set of inclusions, the mean fields $\langle u_i(x)\rangle$ and $\langle\varepsilon_{ij}(x)\rangle$ are also plane shear waves with the wave number β_*, wave normal $\mathbf{n}$ and polarization vector $\mathbf{U} = U\mathbf{m}$,

$$\langle u_i(x)\rangle = m_i U\exp(\mathrm{i}\beta_*\mathbf{n}\cdot\mathbf{x}), \quad \langle\varepsilon_{ij}(x)\rangle = \mathrm{i}\beta_* n_{(i}m_{j)}U\exp(\mathrm{i}\beta_*\mathbf{n}\cdot\mathbf{x}). \tag{3.31}$$

Because (3.28), (3.29) are equations in convolutions the effective external fields $\hat{u}^*_i(x)$ and $\hat{\varepsilon}^*_{ij}(x)$ are also plane waves that may be presented in the forms

$$\hat{u}^*_i(x) = m_i U^u_*\exp(\mathrm{i}\beta_*\mathbf{n}\cdot\mathbf{x}), \tag{3.32}$$

$$\hat{\varepsilon}^*_{ij}(x) = \mathscr{E}^*_{ij}\exp(\mathrm{i}\beta_*\mathbf{n}\cdot\mathbf{x}), \quad \mathscr{E}^*_{ij} = \mathrm{i}\beta_* n_{(i}m_{j)}U^\varepsilon_*. \tag{3.33}$$

Note that amplitudes U^u_* and U^ε_* in these equations do not coincide because, generally speaking, the conditional mean of a derivative does not coincide with the derivative of a conditional mean ($\langle\partial_j u^*_i(x)|x\rangle \neq \partial_j\langle u^*_i(x)|x\rangle$).

Operators Λ^0 and λ^0 in Eqs. (3.16) and (3.17) are defined from the solution of the one particle problem (3.1), (3.2). If we change the fields $u^*_i(x)$ and $\varepsilon^*_{ij}(x)$ in Eqs. (3.1) and (3.2) for their mean values $\hat{u}^*_i(x)$ and $\hat{\varepsilon}^*_{ij}(x)$, the field $u_i(x)$ inside region v with the center at point x_0 takes the form

$$\begin{aligned} u_i(x) &= \lambda^0_{ik}\left[m_k U^u_*\exp(\mathrm{i}\beta_*\mathbf{n}\cdot(\mathbf{x}-\mathbf{x}_0))\exp(\mathrm{i}\beta_*\mathbf{n}\cdot\mathbf{x}_0)\right] = \tilde{\lambda}^0_{ik}(x-x_0)\hat{u}^*_k(x),\\ \tilde{\lambda}^0_{ik}(z) &= \tilde{\lambda}^0_{ik}[\exp(\mathrm{i}\beta_*\mathbf{n}\cdot\mathbf{z})]\exp(-\mathrm{i}\beta_*\mathbf{n}\cdot\mathbf{z}), \end{aligned} \tag{3.34}$$

and the solution of Eq. (3.2) for the field $\varepsilon_{ij}(x)$ in v follows from Eq. (3.34) in the form

$$\begin{aligned} \varepsilon_{ij}(x) &= \left[\partial_{(i}\tilde{\lambda}^0_{j)k}(x-x_0)\right]\frac{in_l}{\beta_*}\left(\mathrm{i}\beta_* n_l m_k U^\varepsilon_*\exp(\mathrm{i}\beta_*\mathbf{n}\cdot\mathbf{x})\right) + \tilde{\lambda}^0_{i)(k}\delta_{l)(j}\hat{\varepsilon}^*_{kl}(x)\\ &= \tilde{\Lambda}^0_{ijkl}(x-x_0)\hat{\varepsilon}^*_{kl}(x), \quad \tilde{\Lambda}^0_{ijkl}(z) = \left[\partial_{(i}\tilde{\lambda}^0_{j)(k}(z)\right]in_{l)}\frac{1}{\beta_*} + \tilde{\lambda}^0_{i)(k}\delta_{l)(j}. \end{aligned} \tag{3.35}$$

Here, functions $\tilde{\lambda}^0_{ik}(z)$ and $\tilde{\Lambda}^0_{ijkl}(z)$ do not depend on the position of the center of the inclusion x_0 and can be found from the solution of the one-particle problem for an inclusion centered at point $x = 0$.

Let us introduce functions $\lambda^u_{ik}(x)$ and $\Lambda^\varepsilon_{ijkl}(x)$ that coincide with functions $\tilde{\lambda}^0_{ik}(x-x^k)$ and $\tilde{\Lambda}^0_{ijkl}(x-x^k)$ inside the region v_k ($k = 1,2,3,\ldots$) and equal to zero in the matrix. Note that functions $\lambda^u_{ik}(x)$ and $\Lambda^\varepsilon_{ijkl}(x)$ compose stationary random fields. Using these functions the fields $u_i(x)$ and $\varepsilon_{ij}(x)$ inside inclusions may be written in the form

$$u_i(x) = \lambda^u(x)\hat{u}^*_k(x), \quad \varepsilon_{ij}(x) = \Lambda^\varepsilon_{ijkl}(x)\hat{\varepsilon}^*_{kl}(x), \quad x\in V. \tag{3.36}$$

Now the action of the operators Λ^0_{ijkl} and λ^0_{ik} on the effective fields $\hat{\varepsilon}^*_{kl}(x)$ and $\hat{u}^*_k(x)$ in Eqs. (3.24)–(3.27) may be presented in the forms

$$\Lambda^0_{ijkl}\hat{\varepsilon}^*_{kl}(x) = \langle \Lambda_{ijkl}\varepsilon^*_{kl}(x)|x\rangle = \langle \Lambda^{\varepsilon}_{ijkl}(x)|x\rangle\hat{\varepsilon}^*_{kl}(x) = \overline{\Lambda}^{\varepsilon}_{ijkl}\hat{\varepsilon}^*_{kl}(x), \tag{3.37}$$

$$\lambda^0_{ik}\hat{u}^*_k(x) = \langle \lambda_{ik}u^*_k(x)|x\rangle = \langle \lambda^u_{ik}(x)|x\rangle\hat{u}^*_k(x) = \bar{\lambda}^u_{ik}\hat{u}^*_k(x), \tag{3.38}$$

$$\overline{\Lambda}^{\varepsilon}_{ijkl} = \frac{1}{\langle v\rangle}\left\langle \int_v \tilde{\Lambda}^0_{ijkl}(x)\,\mathrm{d}x\right\rangle, \quad \bar{\lambda}^u_{ik} = \frac{1}{\langle v\rangle}\left\langle \int_v \tilde{\lambda}^0_{ik}(x)\,\mathrm{d}x\right\rangle. \tag{3.39}$$

Here, $\overline{\Lambda}^{\varepsilon}_{ijkl}$ and $\bar{\lambda}^u_{ik}$ are constant tensors for any homogeneous random field of inclusions (The averaging in these equations is taken over the ensemble distribution of sizes of the inclusions). Hence, operators Λ^0 and λ^0 are products with constant tensors $\overline{\Lambda}^{\varepsilon}_{ijkl}$ and $\bar{\lambda}^u_{ik}$.

Finally, Eq. (3.11) for the mean wave field $\langle u_i(x)\rangle$ in the composite and Eqs. (3.26) and (3.27) for the mean effective fields $\hat{u}^*_i(x)$ and $\hat{\varepsilon}^*_{ij}(x)$ take the forms

$$\langle u_i(x)\rangle = u^0_i(x) + p\int \left[\partial_j G_{ik}(x-x')C^1_{kjmn}\overline{\Lambda}^{\varepsilon}_{mnrs}\hat{\varepsilon}^*_{rs}(x') + \rho_1\omega^2 G_{ik}(x-x')\bar{\lambda}^u_{kl}\hat{u}^*_l(x')\right]\mathrm{d}x, \tag{3.40}$$

$$\hat{u}^*_i(x) = \langle u_i(x)\rangle - p\int \left[\partial_j G_{ik}(x-x')C^1_{kjmn}\overline{\Lambda}^{\varepsilon}_{mnrs}\hat{\varepsilon}^*_{rs}(x') + \rho_1\omega^2 G_{ik}(x-x')\bar{\lambda}^u_{kl}\hat{u}^*_l(x')\right]\Phi(x-x')\,\mathrm{d}x', \tag{3.41}$$

$$\hat{\varepsilon}^*_{ij}(x) = \langle \varepsilon_{ij}(x)\rangle - \int \left[P_{ijkl}(x-x')C^1_{klmn}\overline{\Lambda}^{\varepsilon}_{mnrs}\hat{\varepsilon}^*_{rs}(x') + \rho_1\omega^2\partial_{(i}G_{j)k}(x-x')\bar{\lambda}^u_{kl}\hat{u}^*_l(x')\right]\Phi(x-x')\,\mathrm{d}x'. \tag{3.42}$$

Eqs. (3.40)–(3.42) are equations in convolutions. Therefore, the Fourier transform of these equations gives us the following system of linear algebraic equations with respect to the Fourier transforms of the unknown effective fields

$$\langle u_i(k)\rangle = u^0_i(k) + p\left[ik_j G_{ik}(k)C^1_{kjmn}\overline{\Lambda}^{\varepsilon}_{mnrs}\hat{\varepsilon}^*_{rs}(k) + \rho_1\omega^2 G_{ik}(k)\bar{\lambda}^u_{kl}\hat{u}^*_l(k)\right], \tag{3.43}$$

$$\hat{u}^*_i(k) = \langle u_i(k)\rangle - p\left[\Gamma^{\Phi}_{ijk}(k)C^1_{kjmn}\overline{\Lambda}^{\varepsilon}_{mnrs}\hat{\varepsilon}^*_{rs}(k) + \rho_1\omega^2 G^{\Phi}_{ik}(k)\bar{\lambda}^u_{kl}\hat{u}^*_l(k)\right], \tag{3.44}$$

$$\hat{\varepsilon}^*_{ij}(k) = \langle \varepsilon_{ij}(k)\rangle - \left[P^{\Phi}_{ijkl}(k)C^1_{klmn}\overline{\Lambda}^{\varepsilon}_{mnrs}\hat{\varepsilon}^*_{rs}(x') + \rho_1\omega^2\Gamma^{\Phi}_{(ij)k}(k)\bar{\lambda}^u_{kl}\hat{u}^*_l(x')\right]. \tag{3.45}$$

Here, k is a point of k-space of Fourier transforms, $G_{ik}(k)$ is the Fourier transform of the Green function $G_{ik}(x)$ defined in Eq. (2.2).

$$G_{ik}(k) = \left[L^0_{ik}(k)\right]^{-1}, \quad L^0_{ik}(k) = C^0_{ijkl}k_jk_l - \rho_0\omega^2\delta_{ik}.$$

(We denote Fourier transform of the functions by the same letter with argument k). The functions $G^{\Phi}_{ik}(k)$, $\Gamma^{\Phi}_{ijk}(k), P^{\Phi}_{ijkl}(k)$ in Eqs. (3.43)–(3.45) are

$$G^{\Phi}_{ik}(k) = \int G_{ik}(x)\Phi(x)\mathrm{e}^{\mathbf{ik\cdot x}}\,\mathrm{d}x, \Gamma^{\Phi}_{ijk}(k) = \int \partial_j G_{ik}(x)\Phi(x)\mathrm{e}^{\mathbf{ik\cdot x}}\,\mathrm{d}x,\ P^{\Phi}_{ijkl}(k) = \int P_{ijkl}(x)\Phi(x)\mathrm{e}^{\mathbf{ik\cdot x}}\,\mathrm{d}x. \tag{3.46}$$

Eqs. (3.44) and (3.45) may be written now in the forms

$$T_{ikl}(k)\hat{\varepsilon}^*_{kl}(k) + t_{ik}(k)\hat{u}^*_k(k) = \langle u_i(k)\rangle, \tag{3.47}$$

$$\Pi_{ijkl}(k)\hat{\varepsilon}^*_{kl}(k) + \pi_{ijk}(k)\hat{u}^*_k(k) = \langle \varepsilon_{ij}(k)\rangle, \tag{3.48}$$

where

$$T_{ikl}(k) = p\Gamma^{\Phi}_{ijr}(k)C^{1}_{jrmn}\overline{\Lambda}^{\varepsilon}_{mnkl}, \quad t_{ik}(k) = \delta_{ik} + p\rho_1\omega^2 G^{\Phi}_{ir}(k)\overline{\lambda}^{u}_{rk}, \tag{3.49}$$

$$\Pi_{ijkl}(k) = I_{ijkl} + pP^{\Phi}_{ijrs}(k)C^{1}_{rspq}\overline{\Lambda}^{\varepsilon}_{pqkl}, \quad \pi_{ijk}(k) = p\rho_1\omega^2\Gamma^{\Phi}_{(ij)r}(k)\overline{\lambda}^{u}_{rk}. \tag{3.50}$$

Taking into account Eqs. (3.31)–(3.33) we can write for the Fourier transforms of the functions in Eq. (3.47) and (3.48)

$$\langle u_k(k)\rangle = (2\pi)^3 m_k U\delta(\mathbf{k}+\beta_*\mathbf{n}), \quad \langle\varepsilon_{ij}(k)\rangle = (2\pi)^3 \mathrm{i}\beta_* n_{(i}m_{j)}U\delta(\mathbf{k}+\beta_*\mathbf{n}), \tag{3.51}$$

$$\hat{u}^*_i(k) = (2\pi)^3 m_k U^u_*\delta(\mathbf{k}+\beta_*\mathbf{n}), \quad \hat{\varepsilon}^*_{ij}(k) = (2\pi)^3\mathrm{i}\beta_* n_{(i}m_{j)}U^{\varepsilon}_*\delta(\mathbf{k}+\beta_*\mathbf{n}), \tag{3.52}$$

where $\delta(\mathbf{k})$ is Dirac's delta-function. From these equations and Eqs. (3.47) and (3.48) follows the system connected scalar amplitudes U, U^u_* and U^{ε}_*

$$T(\beta_*)U^{\varepsilon}_* + t(\beta_*)U^u_* = U, \tag{3.53}$$

$$\Pi(\beta_*)U^{\varepsilon}_* + \pi(\beta_*)U^u_* = U. \tag{3.54}$$

Here, scalar coefficients T, t, Π and π are

$$T(\beta_*) = \mathrm{i}\beta_* m_i T_{ikl}(\beta_*)m_k n_l, \quad t(\beta_*) = m_i t_{ik}(\beta_*)m_k, \tag{3.55}$$

$$\Pi(\beta_*) = n_i m_j \Pi_{ijkl}(\beta_*)m_k n_l, \quad \pi(\beta_*) = \frac{1}{\mathrm{i}\beta_*}n_i m_j \pi_{ijk}(\beta_*)m_k, \tag{3.56}$$

and functions $T_{ikl}(\beta_*)$, $t_{ik}(\beta_*)$, $\Pi_{ijkl}(\beta_*)$ and $\pi_{ijk}(\beta_*)$ are defined in Eqs. (3.49) and (3.50), where vector $\mathbf{k}$ has to be replaced with vector $(-\beta_*\mathbf{n})$.

Resolving the system of equations (3.53) and (3.54) with respect to U^{ε}_* and U^u_* we obtain

$$U^{\varepsilon}_* = \frac{1}{\Delta}(t-\pi)U, \quad U^u_* = \frac{1}{\Delta}(\Pi - T)U, \quad \Delta = \Pi t - T\pi. \tag{3.57}$$

Let us multiply both parts of Eq. (3.43) with function $L^0_{ik}(k) = C^0_{ijkl}k_j k_l - \rho_0\omega^2\delta_{ik}$. Taking into account the equations

$$L^0_{ik}(k)G_{kj}(k) = \delta_{ij}, \quad L^0_{ik}(k)u^0_k(k) = 0 \tag{3.58}$$

we obtain

$$L^0_{ik}(k)\langle u_k(k)\rangle - p\left[ik_j C^1_{ijmn}\overline{\Lambda}^{\varepsilon}_{mnrs}\hat{\varepsilon}_{rs}(k) + \rho_1\omega^2\overline{\lambda}^u_{ir}\hat{u}_r(k)\right] = 0. \tag{3.59}$$

Now with the help of Eqs. (3.51), (3.52), (3.57) and (3.58) we can transform Eq. (3.59) into the dispersion equation for the wave number β_* of the mean wave field. This equation may be written in the standard form

$$\beta^2_*\mu_*(\beta_*) - \omega^2\rho_*(\beta_*) = 0, \tag{3.60}$$

$$\mu_*(\beta_*) = \mu_0 + \frac{p\mu_1}{\Delta}\overline{\Lambda}^{\varepsilon}(t-\pi), \quad \overline{\Lambda}^{\varepsilon} = m_i n_j \overline{\Lambda}^{\varepsilon}_{ijkl}m_k n_l, \tag{3.61}$$

$$\rho_*(\beta_*) = \rho_0 + \frac{p\rho_1}{\Delta}\overline{\lambda}^u(\Pi - T), \quad \overline{\lambda}^u = m_i\overline{\lambda}^u_{ik}m_k. \tag{3.62}$$

Eq. (3.60) is the equation for the unknown effective wave number β_* of the mean shear wave propagating through the composite medium. Note that $\overline{\Lambda}^{\varepsilon}$ and $\overline{\lambda}^u$ are functions of the wave number β_*, and these functions have to be found from the solution of the one particle problem (3.1), (3.2). The phase velocity and the attenuation factor of the mean wave field are connected with the wave number β_* by the equations

$$v_* = \frac{\omega}{\mathrm{Re}(\beta_*)}, \quad \gamma = \mathrm{Im}(\beta_*). \tag{3.63}$$

Note that dispersion equation (3.60) may be obtained by the following hypothesis.

Every inclusion in the composite behaves as an isolated one in the original matrix by the action of external fields $u_i^*(x)$ and $\varepsilon_{ij}^*(x)$. These fields are plane waves that are the same for all the inclusions.

This hypothesis is equivalent to the hypotheses H1, H2, H3, but it does not indicate the ways of possible improvements of the obtained solution in the framework of the EFM. For instance, the local external field may be chosen more complex than a plane shear wave, and it changes hypothesis H1. Hypothesis of the quasi-crystalline approximation (H3) may be also changed for a more complex one (see, e.g., Kanaun, 2003), etc.

4. Solution of the one particle problem

Let us consider a shear wave field $\mathbf{u}^*(x)$ with the wave vector $\beta_*\mathbf{e}_3$ and polarization vector $\mathbf{e}_1$ ($\mathbf{e}_i$ is a unit vector of x_i-axis). This field may be presented in the form of a series of spherical vector functions (see Eringen and Suhubi, 1975)

$$\mathbf{u}^*(x) = \mathbf{e}_1 e^{i\beta_* x_3} = \sum_{n=1}^{\infty} \frac{i^n(2n+1)}{n(n+1)} \left[\mathbf{M}_{01n}^1(x) - \frac{i}{\beta_*} \mathbf{N}_{01n}^1(x) \right], \tag{4.1}$$

$$\mathbf{M}_{01n}^1(x) = \mathbf{e}^\theta j_n(\beta_* r) \frac{P_n^1(\cos\theta)}{\sin\theta} \cos\varphi - \mathbf{e}^\varphi j_n(\beta_* r) \frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta} \sin\varphi, \tag{4.2}$$

$$\mathbf{N}_{e1n}^1(x) = \mathbf{e}^r \frac{n(n+1)}{r} j_n(\beta_*) P_n^1(\cos\theta)\cos\varphi + \left[\mathbf{e}^\theta \frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta} \cos\varphi - \mathbf{e}^\varphi \frac{P_n^1(\cos\theta)}{\sin\theta} \sin\varphi \right] \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r}[r j_n(\beta_* r)]. \tag{4.3}$$

Here, $r = |x|$, $\mathbf{e}^r, \mathbf{e}^\theta, \mathbf{e}^\varphi$ are the basic vectors of the spherical coordinate system (r, θ, φ) with polar axis x_3, $j_n(z)$ is the spherical Bessel function and $P_n^1(\cos\theta)$ is the Legendre function of order n.

The one particle problem of the EFM is the problem of diffraction of plane wave (4.1) on a spherical inclusion with elastic moduli λ, μ and density ρ embedded in the matrix material with the dynamic characteristics λ_0, μ_0, ρ_0. If the inclusion has radius a and is centered at point $x = 0$, integral equation (3.1) is equivalent to the following system of partial differential equations

$$\mu\Delta u_i^t + (\lambda+\mu)\partial_i\partial_k u_k^t + \rho\omega^2 u_i^t = \mu_0(\beta_0^2 - \beta_*^2)e_i^1 \exp(i\beta_* x_3), \quad r \leqslant a, \tag{4.4}$$

$$\mu_0\Delta u_i^m + (\lambda_0+\mu_0)\partial_i\partial_k u_k^m + \rho_0\omega^2 u_i^m = \mu_0(\beta_0^2 - \beta_*^2)e_i^1 \exp(i\beta_* x_3), \quad r > a. \tag{4.5}$$

Here, u_i^t is the displacement vector inside the inclusion, u_i^m is the displacement vector in the matrix, Δ is the Laplace operator. These equations differ from the equations of the classical problem of diffraction of a plane monochromatic wave on a spherical inclusion for their right-hand sides are not equal to zero. The latter is the consequence of the fact that the wave number β_* of the effective (local external) field in the one particle problem of the EFM does not coincide with the wave number of the matrix β_0.

The solution of Eqs. (4.4) and (4.5) may be found by the same method as the solution of the classical diffraction problem (see Eringen and Suhubi, 1975). Seeking this solution in the form of the following series

$$\mathbf{u}^t = \sum_{n=1}^{\infty} \left(c_n' \mathbf{L}_{e1n}^{1'} + d_n' \mathbf{M}_{o1n}^{1'} + e_n' \mathbf{N}_{e1n}^{1'} \right) + \zeta^* \mathbf{e}^1 \exp(i\beta_* x_3), \quad \zeta^* = \frac{\mu_0}{\mu} \frac{\beta_0^2 - \beta_*^2}{\beta^2 - \beta_*^2}, \tag{4.6}$$

$$\mathbf{u}^m = \sum_{n=1}^{\infty}\left[c_n\mathbf{L}^3_{e1n} + d_n\mathbf{M}^3_{o1n} + e_n\mathbf{N}^3_{e1n} + \frac{i^n(2n+1)}{n(n+1)}\left(\mathbf{M}^1_{o1n} - \frac{i}{\beta_*}\mathbf{N}^1_{e1n}\right)\right] \tag{4.7}$$

we satisfy the differential equations and the conditions at infinity for the scattered field. Here

$$\mathbf{L}^3_{e1n} = \mathbf{e}^r\frac{\mathrm{d}}{\mathrm{d}r}[h_n(\alpha_0 r)]P^1_n(\cos\theta)\cos\varphi + \left[\mathbf{e}^\theta\frac{\mathrm{d}P^1_n(\cos\theta)}{\mathrm{d}\theta}\cos\varphi - \mathbf{e}^\varphi\frac{P^1_n(\cos\theta)}{\sin\theta}\sin\varphi\right]\frac{h_n(\alpha_0 r)}{r}, \tag{4.8}$$

$\mathbf{M}^3_{o1n}$ and $\mathbf{N}^3_{e1n}$ are obtained from $\mathbf{M}^1_{o1n}$ and $\mathbf{N}^1_{e1n}$ in Eqs. (4.2) and (4.3) by replacing functions $j_n(\beta_* r)$ by $h_n(\beta_0 r)$, $h_n(z)$ is the spherical Hankel function of the first kind. Functions $\mathbf{L}^{1'}_{e1n}, \mathbf{M}^{1'}_{o1n}$ and $\mathbf{N}^{1'}_{e1n}$ are defined by the same equations as functions $\mathbf{L}^1_{e1n}, \mathbf{M}^1_{o1n}$ and $\mathbf{N}^1_{e1n}$ but arguments of Bessel functions should be changed for αr and βr,

$$\alpha^2 = \frac{\omega^2\rho}{\lambda+2\mu}, \quad \beta^2 = \frac{\omega^2\rho}{\mu}. \tag{4.9}$$

Arbitrary constants c_n,d_n,e_n and c'_n, d'_n, e'_n in Eqs. (4.6) and (4.7) have to be found from the conditions on the boundary of the inclusion and the matrix ($r = a$).

$$\mathbf{u}^t(a) = \mathbf{u}^m(a), \quad \mathbf{n}\cdot\boldsymbol{\sigma}^t(a) = \mathbf{n}\cdot\boldsymbol{\sigma}^m(a). \tag{4.10}$$

These conditions give a system of linear algebraic equations for the arbitrary constants in Eqs. (4.6) and (4.7). In details this system is presented in Appendix A.

Tensors $\overline{\lambda}^u_{ik}$ and $\overline{\Lambda}^\varepsilon_{ijkl}$ in Eqs. (3.37)–(3.42) are expressed via integrals from the solution of the one particle problem. Let us begin with Eq. (3.39) for $\overline{\lambda}^u_{ik}$

$$\overline{\lambda}^u_{ik}\hat{u}^*_k = \frac{1}{v}\int_v u^{tu}_i(x)\exp(-\mathrm{i}\beta_*\mathbf{n}\cdot\mathbf{x})\,\mathrm{d}x, \tag{4.11}$$

Here, $u^{tu}_i(x)$ is given by Eq. (4.6) where vector $\mathbf{e}^1$ has to be replaced by $\mathbf{m}U^u_*$. After integration in Eq. (4.11) we obtain

$$\overline{\lambda}^u_{ik}\hat{u}^*_k = h\hat{u}^*_i(x) \quad \text{or} \quad \overline{\lambda}^u_{ik} = h\delta_{ik}, \tag{4.12}$$

$$h = -\frac{3}{2a}\sum_{n=1}^{\infty}(-i)^{n+1}n(n+1)\left\{c'_n\frac{f^1_3(\alpha a)j_n(\beta_* a)}{\beta_* a} - iad'_n g_n(\beta,\beta_*) + e'_n\left[\frac{j_n(\beta a)f^1_4(\beta_* a)}{\beta_* a} + (\beta_* a)g_n(\beta,\beta_*)\right]\right\} + \zeta^*, \tag{4.13}$$

$$g_n(\beta,\beta_*) = \frac{1}{(\beta a)^2 - (\beta_* a)^2}\left[\beta a j_{n+1}(\beta a)j_n(\beta_* a) - \beta_* a j_{n+1}(\beta_* a)j_n(\beta a)\right]. \tag{4.14}$$

From definition (3.37) of tensor $\overline{\Lambda}^\varepsilon_{ijkl}$ follows the equation

$$\overline{\Lambda}^\varepsilon_{ijkl}\hat{\varepsilon}^*_{kl}(x) = \frac{1}{v}\int_v \partial_{(j}u^{t\varepsilon}_{j)}(x)\exp(-\mathrm{i}\beta_*\mathbf{n}\cdot\mathbf{x})\,\mathrm{d}x, \tag{4.15}$$

where $u^{t\varepsilon}_j(x)$ has form in Eq. (4.6) if vector $\mathbf{e}^1$ is replaced with $\mathbf{m}U^\varepsilon_*$.

After calculating the integrals in this equation we obtain

$$\overline{\Lambda}^\varepsilon_{ijkl}\hat{\varepsilon}^*_{kl}(x) = H\hat{\varepsilon}^*_{ij}(x) \text{ or } \overline{\Lambda}^\varepsilon_{ijkl} = HI_{ijkl}, \tag{4.16}$$

where $I_{ijkl} = \delta_{ik}\delta_{jl}|_{(ij)(kl)}$ is a unit four rank tensor,

$$H = -\frac{3}{2a(\beta_*a)^3}\sum_{n=1}^{\infty}(-i)^{n+1}n(n+1)\left(c_n' H_{cn} + id_n' H_{dn} + e_n' H_{en}\right) + \zeta^*, \tag{4.17}$$

$$H_{cn} = 2\left[f_1^1(\alpha a)f_7^1(\beta_* a) + f_3^1(\alpha a)f_8^1(\beta_* a)\right] + (\beta_* a)^2 f_3^1(\alpha a)f_3^1(\beta_* a),$$

$$H_{dn} = -a(\beta_* a)\left[j_n(\beta a)f_7^1(\beta_* a) + (\beta_* a)^2 g_n(\beta,\beta_*)\right],$$

$$H_{en} = 2\left[f_2^1(\beta a)f_7^1(\beta_* a) + f_4^1(\beta a)f_8^1(\beta_* a)\right] + (\beta_* a)^2\left[j_n(\beta a)f_4^1(\beta_* a) + (\beta_* a)^2 g_n(\beta,\beta_*)\right].$$

The coefficients in Eq. (3.57) take the forms

$$T = -p\beta_*\frac{\mu_1}{\mu_0}\Gamma^{\Phi}H, \quad t = 1 + p\frac{\rho_1}{\rho_0}\beta_0^2 G^{\Phi}h, \tag{4.18}$$

$$\Pi = 1 + p\frac{\mu_1}{\mu_0}P^{\Phi}H, \quad \pi = \frac{p}{\beta_*}\frac{\rho_1}{\rho_0}\beta_0^2\Gamma^{\Phi}h, \tag{4.19}$$

where G^{Φ}, Γ^{Φ} and P^{Φ} are the integrals given by formulas (A.1.7)–(A.1.13) in Appendix A.

The obtained equations define all the coefficient in the dispersion equations (3.60)–(3.62), and we can go now to the construction of its solution.

5. Solution of the dispersion equation in the long-wave region

In this section we study the solution of dispersion equation in the long-wave (low-frequency) region where the wave numbers α_0, β_0 and β_* are small ($\alpha_0 a, \beta_0 a, \beta_* a \ll 1$). In this case only main terms in the real and imaginary parts of the coefficients c_n', d_n', e_n' and functions h and H in Eqs. (4.13) and (4.18) should be taken into account. As a result we obtain the coefficients h and H in the forms

$$h \approx \frac{1}{2}(c_1'\alpha + 2e_1'\beta) = 1 + \mathrm{i}(\beta_0 a)^3\frac{\rho_1}{9\rho_0}(2+\eta_0^2), \quad \eta_0 = \frac{\alpha_0}{\beta_0}, \tag{5.1}$$

$$H \approx -\frac{i}{15\beta_0}\left(c_2'\alpha^2 + 3e_2'\beta^2\right) = H_s - \mathrm{i}(\beta_0 a)^3 H_\omega, \tag{5.2}$$

$$H_s = \left[1 + \frac{2\mu_1}{15\mu_0}(3+2\eta_0^2)\right]^{-1}, \quad H_\omega = \frac{2}{45}(3+2\eta_0^2)\frac{\mu_1}{\mu_0}(H_s)^2. \tag{5.3}$$

Let us consider the coefficients T, t, Π and π in Eqs. (4.18) and (4.19) in the long-wave limit. With the accuracy $(\beta_0 a)^3$ we obtain

$$T = \pi = 0, \tag{5.4}$$

and the asymptotics of integrals G^{Φ}, P^{Φ} in Eqs. (A.1.7)–(A.1.9) and coefficients Π and t take the forms

$$G^{\Phi} \approx \frac{1}{3}(2+\eta_0^2)J^0 + \frac{\mathrm{i}\beta_0}{3}(2+\eta_0^3)J, \quad J^0 = \int_0^{\infty}\Phi(r)r\,\mathrm{d}r, \quad J = \int_0^{\infty}\Phi(r)r^2\,\mathrm{d}r, \tag{5.5}$$

$$P^{\Phi} \approx -\left(P_s + \mathrm{i}\beta_0^3 J P_\omega\right), \quad P_s = \frac{2(3+2\eta_0^2)}{15}, \quad P_\omega = \frac{2(3+2\eta_0^5)}{15}, \tag{5.6}$$

$$\Pi \approx 1 - p\frac{\mu_1}{\mu_0}\Big((P_s + \mathrm{i}\beta_0^3 J P_\omega)H_s - \mathrm{i}(\beta_0 a)^3 P_s H_\omega\Big), \quad \mu_1 = \mu - \mu_0, \tag{5.7}$$

$$t \approx 1 + \mathrm{i}p\frac{\rho_1}{3\rho_0}(2+\eta_0^3)\beta_0^3 J, \quad \rho_1 = \rho - \rho_0. \tag{5.8}$$

The effective shear modulus μ_* and effective density ρ_* in dispersion equation (3.60) in the long-wave region take the forms

$$\mu_* = \mu_s - \mathrm{i}\beta_0^3 pvf\mu_\omega, \quad \rho^* = \rho_s + \mathrm{i}\beta_0^3 pvf\rho_\omega, \tag{5.9}$$

$$v = \frac{4}{3}\pi a^3, \quad f = 1 - \frac{4\pi}{v}pJ, \tag{5.10}$$

where μ_s is the static effective shear modulus of the composite ($\beta_0 \to 0$), and ρ_s is a "static" density.

$$\mu_s = \mu_0 + p\mu_R, \quad \mu_R = \frac{\mu_1 H_s}{1 - p\mu_1 H_s P_s} = \left[\frac{1}{\mu_1} + (1-p)\frac{2(3+2\eta_0^2)}{15\mu_0}\right]^{-1}, \quad \rho_s = \rho_0 + p\rho_1. \tag{5.11}$$

The factors μ_ω and ρ_ω in the imaginary parts of μ_* and ρ_* in Eq. (5.9) take the following forms

$$\mu_\omega = \mu_R^2\frac{3+2\eta_0^5}{30\pi\mu_0}, \quad \rho_\omega = \rho_1^2\frac{2+\eta_0^3}{12\pi\rho_0}. \tag{5.12}$$

The dispersion equation (3.60) in the long-wave is presented in the form

$$\beta_*^2 = \beta_s^2\left[1 + \mathrm{i}\beta_0^3 pvf\left(\frac{\mu_\omega}{\mu_s} + \frac{\rho_\omega}{\rho_s}\right)\right], \quad \beta_s^2 = \frac{\omega^2\rho_s}{\mu_s}. \tag{5.13}$$

Thus, effective wave number β_* is

$$\beta_* = \beta_s + \mathrm{i}\gamma, \tag{5.14}$$

where the attenuation factor γ takes the form

$$\gamma = \frac{pf(\beta_0 a)^4}{18a}\left(\frac{\beta_s}{\beta_0}\right)\left[\frac{2}{5}\frac{\mu_R^2}{\mu_0\mu_s}(3+2\eta_0^5) + \frac{\rho_1^2}{\rho_0\rho_s}(2+\eta_0^3)\right]. \tag{5.15}$$

Eqs. (5.9)–(5.15) for the effective parameters of the composite coincide with the results of Willis (1980) obtained by the solution of the problem of wave propagation in particulate composites in the long-wave region.

In Figs. 1 and 2, approximation (5.11) for the effective shear modulus μ_s of the composites is compared with the numerical computation of these moduli presented in Segurado and Llorca (2002). Solid line in Fig. 1 is the dependence of μ_s on volume concentration of inclusions p that corresponds to Eq. (5.11) in the case of absolutely rigid inclusions, the dashed line presents the numerical results of Segurado and Llorca (2002). The same dependencies for the material with spherical pores are presented in Fig. 2. The numerical results of Segurado and Llorca (2002) were obtained by finite element technique applied to the solution of the elasticity problem for the characteristic volume of the composite material. The number of inclusion inside the characteristic volume and the sizes of finite elements used by the calculations in this work allow us to consider the obtained result as an exact solution of the homogenization problem or being very close to such a solution. Thus, the graphs in Figs. 1 and 2 may be interpreted as a comparison of the predictions of the EFM and the exact values of the elastic shear moduli of the composites with spherical inclusions.

Attenuation factor γ of the mean wave field in the long-wave region is proportional to factor $(\beta_0 a)^4$ (Rayleigh scattering) and to structural factor f

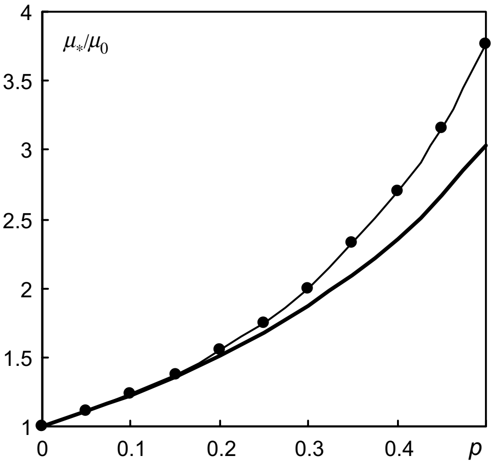

Fig. 1. The dependence of the effective shear modulus of the composite with rigid inclusions on the volume concentration of the latter. Solid line is the prediction of the EFM, line with dots is the results of numerical calculations of Segurado and Llorca (2002).

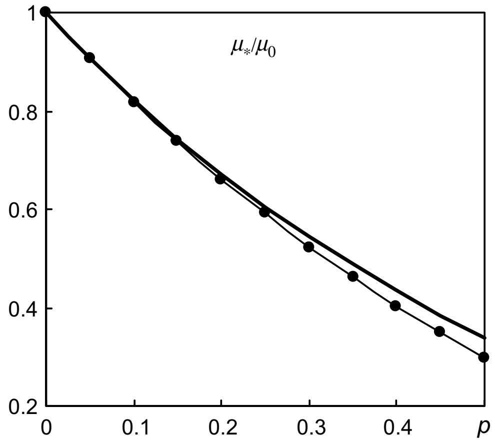

Fig. 2. The dependence of the effective shear modulus of the medium with spherical pores on the volume concentration of the latter. Solid line is the prediction of the EFM, line with dots is the results of numerical calculations of Segurado and Llorca (2002).

$$f = 1 - \frac{4\pi}{v} pJ = 1 - 3p \int_0^\infty \Phi(\zeta)\zeta^2 \, d\zeta, \quad \zeta = \frac{r}{a}. \tag{5.16}$$

Factor f as well as function $\Phi(r)$ depends only on geometrical properties of the random field of inclusions. It is shown in Appendix B that factor f is non-negative ($f \geqslant 0$) for any realizable correlation function of a spatially homogeneous random set of inclusions.

Further we use Percus–Yevick correlation function $\psi(r)$, ($r = |x|$) for the construction of function $\Phi(r)$ and structural factor f. The value of $\psi(r)$ is the probability density to find a center of an inclusion at point x if the center of other inclusion is situated in the origin ($x = 0$). An explicit equation for the Percus–Yevick correlation function $\psi(r)$ is presented in Wertheim (1963). The behavior of this functions for $p = 0.1, 0.3, 0.5$ is shown in Fig. 3. Function $\Phi(r)$ is connected with function $\psi(r)$ by the equation

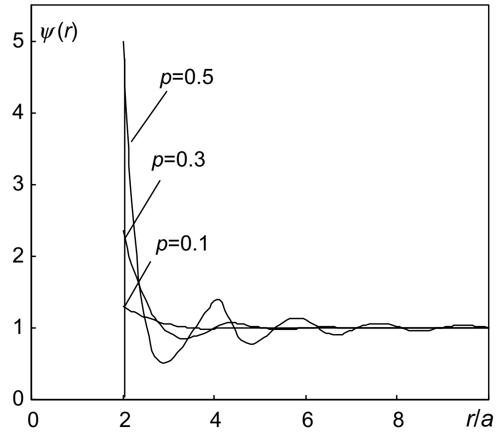

Fig. 3. The graphs of Percus–Yevick correlation function $\psi(\zeta)$ for various volume concentrations of inclusions.

$$\Phi(r) = 1 - \frac{1}{v^2}\int v_0(x)\,\mathrm{d}x \int \psi(|x-y|)v_0(y-z)\,\mathrm{d}y, \quad r = |z|. \tag{5.17}$$

Here, $v_0(x)$ is the characteristic function of a spherical region of a unit radius centered at point $x = 0$. The double integral over 3D-space in this formula may be reduced to a single (one-dimensional) integral (see Kanaun and Jeulin, 1997). The graphs of function $\Phi(r)$ for various values of $p(p = 0.1, 0.3, 0.5)$ are presented in Fig. 4.

After substituting $\Phi(r)$ from Eq. (5.17) into Eq. (5.16) for the factor f we obtain the following equation

$$f = 1 - p\left[8 + 3\int_2^\infty (1-\psi(\zeta))\zeta^2\,\mathrm{d}\zeta\right], \quad \zeta = \frac{r}{a}. \tag{5.18}$$

For Percus–Yevick correlation function $\psi(r)$ structural factor f takes the form (see Twersky, 1975; Willis, 1980)

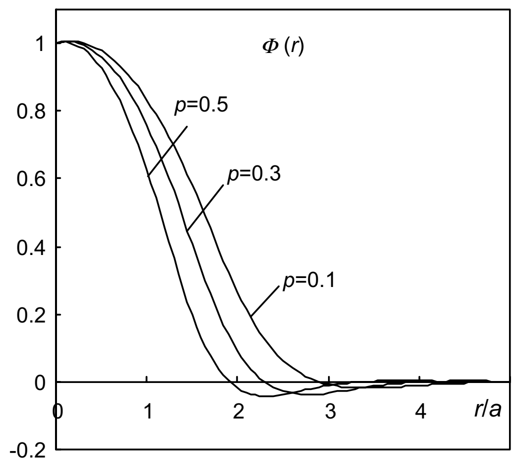

Fig. 4. The graphs of specific correlation function $\Phi(\zeta)$ corresponding to the Percus–Yevick correlation function ψ.

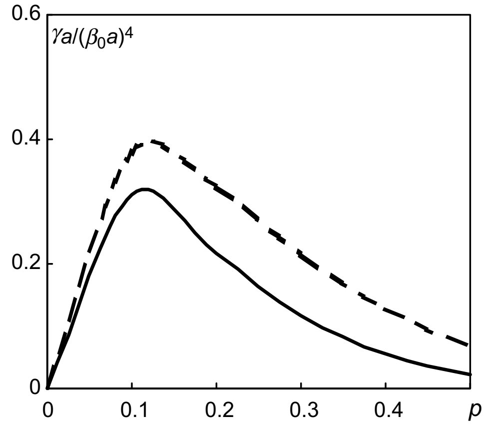

Fig. 5. The dependence of attenuation factor γ in the long-wave region on the volume concentrations of inclusions p. Solid line corresponds to the medium with rigid inclusions, dashed line corresponds to the medium with spherical pores.

$$f = \frac{(1-p)^4}{(1+2p)^2}. \tag{5.19}$$

The dependences of the attenuation factor $\gamma a/(\beta_0 a)^4$ calculated from Eq. (5.15) on the volume concentration of inclusions p are presented in Fig. 5. Solid line in this figure corresponds to rigid inclusions, the dashed line is the case of the porous medium.

6. Solution of the dispersion equation in the short-wave limit

Let us consider the solution of the dispersion equation of the EFM in the short-wave limit. In this case $\omega, \alpha_0, \beta_0 \to \infty$ and as it follows from Eqs. (4.13) and (4.17) $h, H \to 0$. It is known that in the short-wave limit the effective wave number of the mean wave field has the form (Waterman and Truel, 1961; Bussemer et al., 1991; Kanaun, 2000)

$$\beta_* = \mathrm{Re}\,\beta_* + \mathrm{i}\gamma, \tag{6.1}$$

where γ does not depend on β_0, and $\mathrm{Re}\,\beta_* = \mathrm{O}(\beta_0)$. It is possible to show that integrals G^Φ, Γ^Φ and P^Φ in Eq. (A.1.7)–(A.1.9) in the short-wave limit take the forms

$$\lim_{\beta_0\to\infty} \beta_0^2 G^\Phi(\beta_*) = \beta_0^2 \int_0^\infty \mathrm{e}^{\mathrm{i}\beta_0 r} j_0(\beta_* r)\Phi(r)\,\mathrm{d}r = \frac{1}{2}\mathrm{i}\beta_0 a I(\gamma a), \tag{6.2}$$

$$\lim_{\beta_0\to\infty} P^\Phi(\beta_*) = \lim_{\beta_0\to\infty} \beta_0 \Gamma^\Phi(\beta_*) = \mathrm{i}\beta_0^2 \int_0^\infty \mathrm{e}^{\mathrm{i}\beta_0 r} j_1(\beta_* r)\Phi(r)\,\mathrm{d}r = \frac{1}{2} i\beta_0 a I(\gamma a), \tag{6.3}$$

$$I(\gamma a) = \int_0^\infty \mathrm{e}^{\gamma a \zeta}\Phi(\zeta)d\zeta, \quad \zeta = \frac{r}{a}. \tag{6.4}$$

Here, the limit form of β_* (6.1) together with asymptotic formulas $j_0(\beta_0 r) \sim \sin(\beta_* r)/\beta_* r$, $j_1(\beta_* r) \sim -\cos(\beta_* r)/\beta_* r$ for large β_* were used. Taking into account these relations we obtain from Eqs. (3.61) and (3.62)

$$\mu_* = \mu_0 + \frac{p\mu_1}{\Delta}H, \quad \rho_* = \rho_0 + \frac{p\rho_1}{\Delta}h \tag{6.5}$$

$$\Delta = 1 + \frac{1}{2}ip\beta_0 I(\gamma)\Lambda, \quad \Lambda = \left(\frac{\rho_1}{\rho_0}h - \frac{\mu_1}{\mu_0}H\right). \tag{6.6}$$

Note that $\Lambda/\Delta \to 0$ when $\omega \to \infty$.

It follows from Eqs. (3.60), (6.5) and (6.6) that in the short-wave limit the equation for the effective wave number β_* may be written as

$$\beta_*^2 = \frac{\rho_0\omega^2}{\mu_0}\left(1 + p\frac{\Lambda}{\Delta}\right), \quad \text{or} \quad \beta_* = \beta_0\left(1 + p\frac{\Lambda}{2\Delta}\right). \tag{6.7}$$

Let us consider the short-wave limit of function Λ in Eq. (6.6). It follows from Eqs. (4.13) and (4.17) that for large values of ω (or β_0) the equation for $\beta_0\Lambda$ takes the form

$$\beta_0\Lambda = \beta_0\left(\frac{\rho_1}{\rho_0}h_0 - \frac{\mu_1}{\mu_0}H_0\right) + \beta_0\left(\frac{\rho_1}{\rho_0} - \frac{\mu_1}{\mu_0}\right)\frac{\mu_0}{\mu}\frac{\beta_0^2 - \beta_*^2}{\beta^2 - \beta_*^2}. \tag{6.8}$$

Here, h_0 and H_0 are the same as in Eq. (A1.18) (see Appendix A). As it follows from Eq. (A1.19) the limit value of the first term in the right-hand side of Eq. (6.8) when $\omega \to \infty$ is equal to $3i/2a$, and the limit of the last term in this equation is $-2i\gamma$. As a result, we obtain the short-wave limits of functions $\beta_0\Lambda$ and Δ in Eq. (6.7) in the form

$$\lim_{\omega\to\infty} \beta_0\Lambda = \frac{2i}{a}\left(\frac{3}{4} - \gamma a\right), \quad \lim_{\omega\to\infty} \Delta = 1 - p\left(\frac{3}{4} - \gamma a\right)I(\gamma a). \tag{6.9}$$

Thus, in the short-wave limit Eq. (6.7) takes the form

$$\beta_* a = \beta_0 a + p\frac{i\left(\frac{3}{4} - \gamma a\right)}{1 - p\left(\frac{3}{4} - \gamma a\right)I(\gamma a)}. \tag{6.10}$$

It follows from this equation that the phase velocity of the mean wave field coincides with the velocity of shear waves in the matrix material

$$v_* = \frac{\omega}{\mathrm{Re}(\beta_*)} = \frac{\omega}{\beta_0} = v_0, \tag{6.11}$$

and from Eqs. (6.1) and (6.10) we obtain that the short-wave limit $\bar{\gamma}$ of the attenuation factor γ is

$$\bar{\gamma} a = \mathrm{Im}(\beta_* a) = p\left(\frac{3}{4} - \bar{\gamma} a\right)\left[1 - p\left(\frac{3}{4} - \bar{\gamma} a\right)I(\bar{\gamma} a)\right]^{-1}. \tag{6.12}$$

(6.12) is in fact the equation for the short-wave limit $\bar{\gamma}$ of the attenuation factor. For small volume concentrations of inclusions ($p \ll 1$) the value of $\bar{\gamma}$ is

$$\bar{\gamma} a = \frac{3}{4}p.$$

The graph of the function $\bar{\gamma} a$ for the Percus–Yevick correlation function is presented in Fig. 6.

Thus, the limit value of the attenuation factor depends neither on the frequency of the incident field nor on the properties of the inclusions and the matrix, and $\bar{\gamma} a$ is only a function of the inclusion volume concentration p and their distribution in space (for large p it depends on the correlation function $\Phi(r)$ via integral $I(\gamma a)$ in Eq. (6.4)). This result may be interpreted as follows. In the short-wave limit the geometrical optics interpretation may be used for the description of the mean wave field in the composite. This field

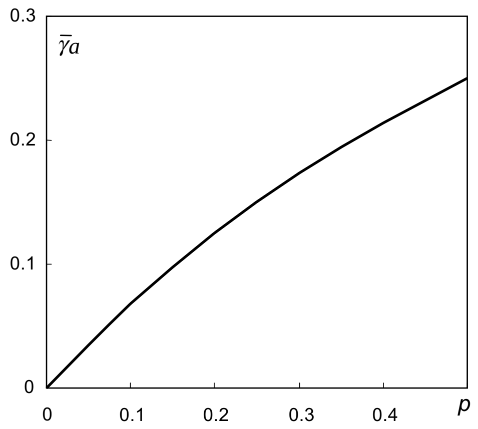

Fig. 6. The dependence of the short-wave limit $\bar{\gamma}$ of the attenuation factor of the mean wave field on the volume concentration of inclusions p.

may be considered as a set of independent beams propagating through the medium. Because of existence of a continuous component (matrix) the phase velocity of the mean field should coincide with the wave velocity in the matrix. The attenuation factor γ in the short-wave limit does not depend on the frequency and properties of inclusions and is only a function of a number of scatterers on a unit length (for electromagnetic waves see similar results in Bussemer et al. (1991) and Kanaun (2000)). The latter is the consequence of the extinction paradox for the value of the total scattering cross section of the inclusion in the short-wave limit (see Appendix A).

7. Numerical solution of the dispersion equation

Numerical analysis of dispersion equation (3.60) discovered several branches of its solutions. Three dispersion curves (different branches of the solution of the dispersion equation in the region $0 < \beta_* a, \beta_0 a < 3$) for the medium with hard and heavy inclusions ($\rho/\rho_0 = 10, E/E_0 = 50, v = 0.3, v_0 = 0.4, p = 0.3$) are presented in Fig. 7. The dependencies of the real parts of the effective wave number on the wave number of the matrix ($\mathrm{Re}\beta_*(\beta_0)$) are in the left figure, and the dependences of the imaginary parts $\mathrm{Im}\beta_*(\beta_0)$ are in the right figure. In the long- and short-wave regions, the behavior of branch 1 coincides with the asymptotic solutions obtained in the previous sections. This branch may be called the acoustical (quasiacoustical) branch. The second branch (2) is lower than the acoustical branch and starts with a finite values of frequency ($\beta_0 a \approx 0.7$); this branch may be called the optical (quasioptical) branch. The third branch (3) is higher than the acoustical branch and starts with the point that correspond to the root of Eq. (3.60) for $\omega = 0$. The existence of such non-trivial roots of the dispersion equations is typical for a medium with microstructure (see Kunin, 1980, pp. 51–58; 1983, pp. 33–37). For the quasi-continuum models presented in Kunin (1980, 1983) these roots are imaginary numbers. The non-trivial root of Eq. (3.60) for $\omega = 0$ turned to be a complex number with a non-zero real part. The attenuation factors γa of the waves that correspond to branches 2 and 3 are several orders of magnitude more than the attenuation factors of the acoustic waves and are about 2 along these branches. Thus, these waves are practically attenuate on the length of the diameter of inclusions, and it is difficult to observe these waves in experiments. Note that the quasi-optical branches were found out in some suspensions of elastic particles (see Sheng et al., 1994).

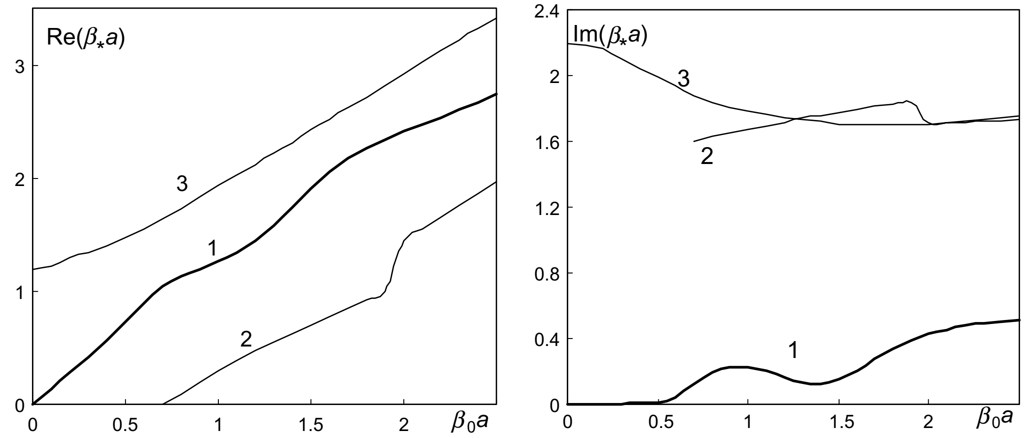

Fig. 7. The dependence of real $\mathrm{Re}(k_*a)$ and imaginary $\mathrm{Im}(k_*a)$ parts of the wave number of the mean wave field on the wave number in the matrix material β_0 for the medium with hard and heavy inclusions. 1 is the quasi-acoustical branch of the solution of the dispersion equation (3.60), 2 is quasi-optical branch, and 3 is the branch typical for a non-local medium.

The following iterative procedure was used for the numerical solution of dispersion equation (3.60)

$$\beta_*^{(n)} = \beta_*^{(n-1)} - \varepsilon\left[\beta_*^{(n-1)} - \omega\sqrt{\frac{\rho_*\left(\beta_*^{(n-1)}\right)}{\mu_*\left(\beta_*^{(n-1)}\right)}}\right]. \tag{7.1}$$

Here, index n corresponds to the number of the iteration, parameter ε ($|\varepsilon| < 1$) is to be chosen for convergence of the iterative process. Functions $\rho_*(\beta_*)$ and $\mu_*(\beta_*)$ are defined in Eq. (3.61), (3.62). As a "zero" iteration a simple additive law of the dependence of the effective parameters of the composite on the microstructure was assumed. Such an iterative procedure was used for the construction of the acoustical branch of dispersion equation (3.60). The solutions corresponded to the second and third branches were found by seeking direct minima of the function

$$F(\beta_*) = \left|\beta_*^{(n-1)} - \omega\sqrt{\frac{\rho_*\left(\beta_*^{(n-1)}\right)}{\mu_*\left(\beta_*^{(n-1)}\right)}}\right| \tag{7.2}$$

in the complex plane ($\mathrm{Re}\beta_*, \mathrm{Im}\beta_*$). Note that in the region of middle wave lengths and for high volume concentrations of inclusions, the effective wave numbers of these three branches turn to be close, and the iterative procedure (7.1) may converge to a solution corresponded to branch 2 or 3. In this region these three branches should be carefully separated.

The results of calculation of the phase velocities and attenuation factors of shear waves that correspond to the acoustical branch of the solutions of the dispersion equation are presented in Figs. 8 and 9. The cases of composites with hard and heavy inclusions ($\rho/\rho_0 = 10$, $E/E_0 = 50$, $v = 0.3$, $v_0 = 0.4$, and E,v and E_0,v_0 are Young moduli and Poisson ratios of the inclusions and the matrix) and volume concentrations of inclusions $p = 0.1$ and $p = 0.3$ are in Fig. 8. The considered region of wave number β_0 covers the long-, middle- and short-regions ($0 < \beta_0 a < 100$, logarithmic scale is used in Fig. 8). More detailed behavior of these dependences in the region ($0 < \beta_0 a < 3$) is shown in Fig. 9, where non-logarithmic scale is used. The dashed horizontal lines in these figures are the short-wave asymptotics of the velocities and attenuation factors of waves obtained in Section 6 (Eqs. (6.11) and (6.12)).

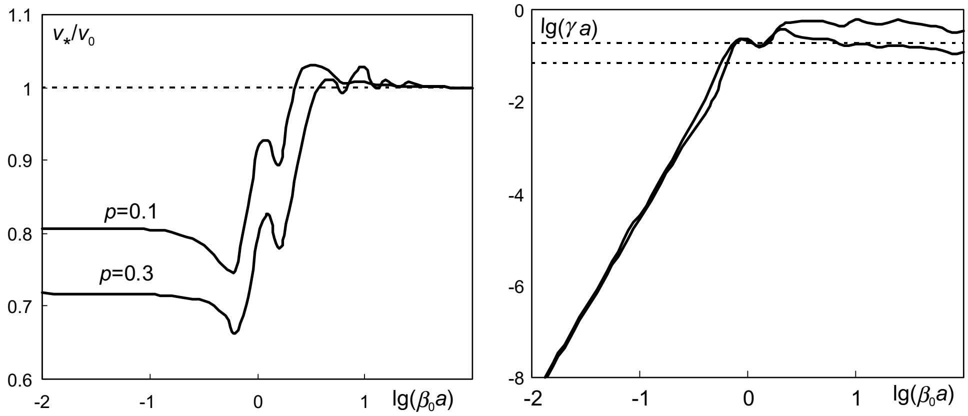

Fig. 8. The dependences of relative velocity v_*/v_0 (v_0 is the velocity of shear waves in the matrix) and attenuation factor γ of the mean wave field on the frequency of the incident field (wave number of the matrix material β_0) for the medium with heavy and hard inclusions ($\rho/\rho_0 = 10, E/E_0 = 50, v = 0.3, v_0 = 0.4$).

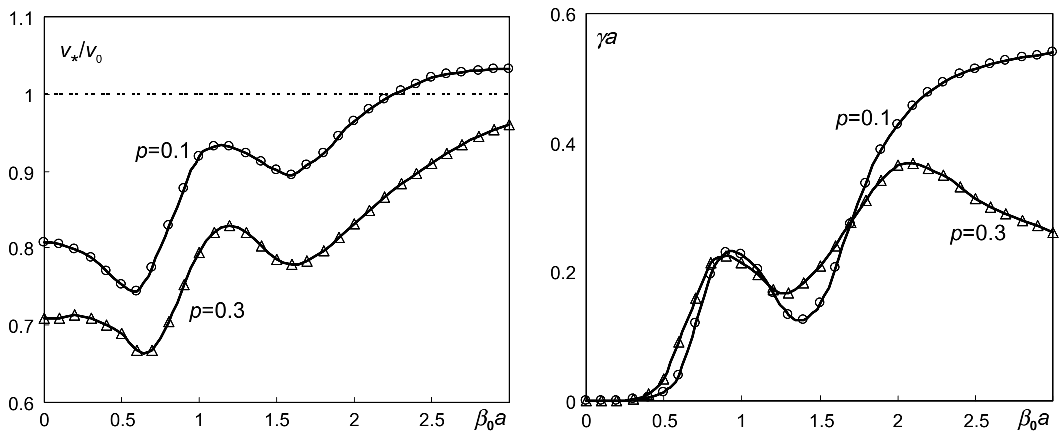

Fig. 9. The same dependences as in Fig. 8 in the non-logarithmic scale.

The results of calculation of the phase velocities and attenuation factors of shear waves for the composite with soft and light inclusions ($\rho/\rho_0 = 0.1, E/E_0 = 0.02, v = 0.3, v_0 = 0.4$) are presented in Fig. 10. The graphs in this figure are constructed with the step 0.25 in the logarithmic scale and don't reflect small-scale oscillations that are essential in this case. Detailed dependences of the velocities and attenuation factors on the mean wave field on frequency ($\beta_0 a$) in the region $0 < \beta_0 a < 4$ are presented in Fig. 11. Note that in some region of middle wave lengths ($0.6 < \beta_0 a < 1.8$ for $p = 0.3$) the imaginary part of the solutions of dispersion equation (3.60) (attenuation factor) turns to be negative (the dashed part of the curve in the right-hand side of Fig. 10 corresponds to negative values of the attenuation factors). In this region the method overestimate interactions between inclusions, and its predictions for the attenuation factors become physically not correct. For small volume concentrations of inclusions ($p < 0.1$) the attenuation factor is positive. For hard and heavy inclusions the method predicts positive values of the attenuation factors for all frequencies of the incident field and volume concentrations of inclusions.

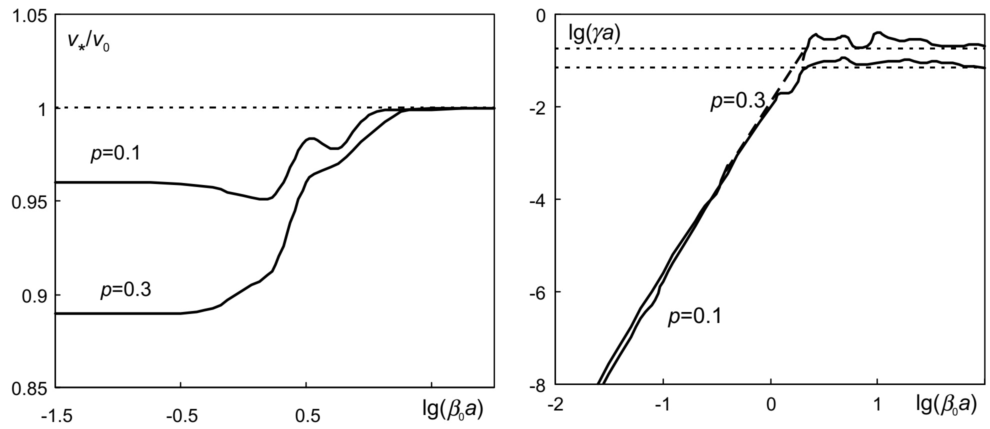

Fig. 10. The dependences of relative velocity v_*/v_0 and attenuation factor γ of the mean wave field on the wave number of the matrix material β_0 for the medium with light and soft inclusions ($\rho/\rho_0 = 0.1, E/E_0 = 0.02, v = 0.3, v_0 = 0.4$).

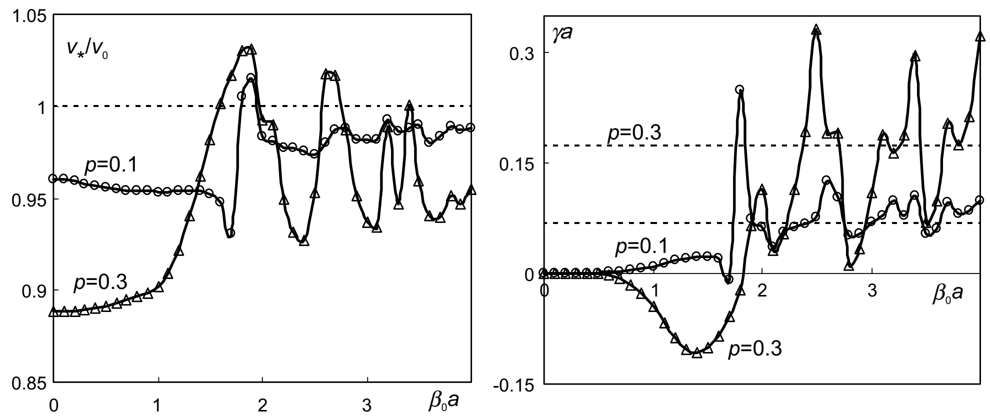

Fig. 11. The same dependences as in Fig. 10 in the non-logarithmic scale.

8. Conclusion

The version of the EFM developed in this work allows us to obtain the dispersion equation for the wave numbers of the mean elastic shear wave fields in two phase particulate composites. This dispersion equation serves for all frequencies of the incident field, arbitrary properties of the phases and their volume concentrations.

In the long-wave region, the method gives physically correct values of the velocities and attenuation factors of the mean wave field in the composites. Its predictions of the effective shear moduli of particulate composites correspond to the numerical calculations of the latter in Segurado and Llorca (2002) and to experimental data (see Kanaun and Levin, 1994). The error of the EFM in this region is essential if the inclusions are much harder than the matrix, and its volume concentration is more than 0.3 (see Fig. 1).

In the middle wave region, the method gives physically reasonable values of the phase velocities of the mean wave fields but it predicts negative values of attenuation factors for the composites with high volume concentrations of the inclusions that are much softer and lighter than the matrix. Apparently, the picture of the detailed wave field in the middle wave region is very complex and cannot be described by relatively simple hypotheses of the EFM.

In the short-wave region the method gives physically correct results for all types of inclusions.

In the framework of the method, it is possible to change its main hypotheses for more appropriate ones. In particular, the effective field that acts on every inclusion in the composite may be taken more complex than a plane wave as it was assumed in hypothesis H1. Hypothesis H3 of quasi-crystalline approximation (see Eqs. (3.24) and (3.25)) may be also changed for the following

$$\langle \Lambda_{ijkl}\varepsilon^*_{kl}(x)|x,x',x''\rangle = \langle \Lambda_{ijkl}\varepsilon^*_{kl}(x)|x,x'\rangle, \tag{8.1}$$

$$\langle \lambda_{ik}u^*_k(x)|x,x',x''\rangle = \langle \lambda_{ik}u^*_k(x)|x,x'\rangle. \tag{8.2}$$

This hypothesis closes the chain of equations for the many point conditional means of the local external field on the second step (see details in Kanaun, 2003).

Comparison of the EFM with other self-consistent methods (various versions of the effective medium method (EMM)) shows that these methods give close results in the long-wave region and for small volume concentrations of inclusions, but in the middle and short-wave regions the predictions of the EFM and EMM may deviate essentially. These predictions are also different for the composites with high volume concentrations of contrast inclusions (see the comparison of the EFM and EMM predictions in the case of shear wave propagation in fiber composites in Kanaun and Levin (2003), the analysis of various versions of the EMM is presented in Kanaun et al. (2004)). The main advantage of the EFM in comparison with various versions of the EMM is the possibility to take into account the influence of peculiarities in spatial distributions of inclusions on the mean wave field in composites. For electromagnetic waves, such influence was studied in Kanaun and Jeulin (1999) and Kanaun (2000). It was shown in the last work that the existence of photonic gaps in the frequency region in composite materials with regular microstructures may be described by the considered version of the EFM.

Appendix A. One-particle problem of the EFM for shear wave propagation

A.1. Equations for the constants in the solution of the one particle problem

The constants in the solution (4.6) and (4.7) of the one particle problem are to be found from the boundary conditions (4.10). These conditions give the following system of linear algebraic equation for the constants

$$\mathbf{L}_n\begin{pmatrix} c_n \\ e_n \end{pmatrix} - \mathbf{L}'_n\begin{pmatrix} c'_n \\ e'_n \end{pmatrix} = i^{n+1}\frac{2n+1}{n(n+1)}\frac{1}{\beta_*}(1-\zeta^*)\begin{pmatrix} f^1_2(\beta_*a) \\ f^1_4(\beta_*a) \end{pmatrix} \tag{A.1.1}$$

$$\mathbf{M}_n\begin{pmatrix} c_n \\ e_n \end{pmatrix} - \frac{\mu}{\mu_*}\mathbf{M}'_n\begin{pmatrix} c'_n \\ e'_n \end{pmatrix} = i^{n+1}\frac{2n+1}{n(n+1)}\frac{1}{\beta_*}\left(1-\frac{\mu}{\mu_0}\zeta^*\right)\begin{pmatrix} f^1_6(\beta_*a) \\ f^1_8(\beta_*a) \end{pmatrix}, \tag{A.1.2}$$

where the matrices $\mathbf{L}_n$ and $\mathbf{M}_n$ are

$$\mathbf{L}_n = \begin{pmatrix} f^2_1(k_*a) & f^2_2(\kappa_*a) \\ f^2_3(k_*a) & f^2_4(\kappa_*a) \end{pmatrix}, \quad \mathbf{M}_n = \begin{pmatrix} f^2_5(k_*a) & f^2_6(\kappa_*a) \\ f^2_7(k_*a) & f^2_8(\kappa_*a) \end{pmatrix}. \tag{A.1.3}$$

$\mathbf{L}'_n$ and $\mathbf{M}'_n$ in Eqs. (A.1.1) and (A.1.2) have forms (A.1.3) if radial function $f^2_m(\alpha_0a), f^2_m(\beta_0a)$ are replaced with $f^1_m(\alpha a), f^1_m(\beta a)$. The system for the constants d_n and d'_n has the form

$$h_n(\beta_0 a)d_n - j_n(\beta a)d'_n = -i^n \frac{2n+1}{n(n+1)}(1-\zeta^*)j_n(\beta_* a), \tag{A.1.4}$$

$$f_6^2(\beta_0 a)d_n - \frac{\mu}{\mu_0}f_6^1(\beta a)d'_n = -i^n \frac{2n+1}{n(n+1)}\left(1-\frac{\mu}{\mu_0}\zeta^*\right)f_6^1(\beta_* a). \tag{A.1.5}$$

The radial functions $f_m^i(qr)$, $(m = 1, 2, \ldots, 9,\ i = 1, 2)$ in (A.1.1)–(A.1.5) have forms

$$\begin{aligned}
&f_1^i(\alpha r) = ny_n^i(\alpha r) - \alpha r y_{n+1}^i(\alpha r), \quad f_2^i(\beta r) = n(n+1)y_n^i(\beta r),\\
&f_3^i(\alpha r) = y_n^i(\alpha r), \quad f_4^i(\beta r) = (n+1)y_n^i(\beta r) - \beta r y_{n+1}^i(\beta r),\\
&f_5^i(\alpha r) = \left(n^2 - n - \frac{(\beta r)^2}{2}\right)y_n^i(\alpha r) + 2\alpha r y_{n+1}^i(\alpha r),\\
&f_6^i(\beta r) = n(n+1)\left[(n-1)y_n^i(\beta r) - \beta r y_{n+1}^i(\beta r)\right],\\
&f_7^i(\alpha r) = (n-1)y_n^i(\alpha r) - \alpha r y_{n+1}^i(\alpha r),\\
&f_8^i(\beta r) = \left(n^2 - 1 - \frac{(\beta r)^2}{2}\right)y_n^i(\beta r) + \beta r y_{n+1}^i(\beta r).
\end{aligned} \tag{A.1.6}$$

In these equations wave numbers α and β are without indices for the fields inside the inclusion, they have index "0" for the fields in the matrix and "$*$" for the medium with the effective properties. If $i = 1$ functions $y_n^1(z)$ are spherical Bessel functions $j_n(z)$, for $i = 2$ these functions are Hankel functions $h_n(z)$.

A.2. Integrals in Eqs. (4.18) and (4.19)

The integrals in Eqs. (4.18) and (4.19) depend on the specific correlation function $\Phi(r)$ and have forms

$$G^{\Phi} = \int_0^{\infty}\left[G_1(r)j_0(\beta_* r) + G_2(r)\frac{j_1(\beta_* r)}{\beta_* r}\right]\Phi(r)r\,\mathrm{d}r, \tag{A.1.7}$$

$$\Gamma^{\Phi} = -\int_0^{\infty}\left\{G_2(r)\left[j_1(\beta_* r) - \frac{4j_2(\beta_* r)}{\beta_* r}\right] + G_3(r)j_1(\beta_* r) + G_4(r)\frac{2j_2(\beta_* r)}{\beta_* r}\right\}\Phi(r)\,\mathrm{d}r, \tag{A.1.8}$$

$$\begin{aligned}
P^{\Phi} = -\int_0^{\infty}&\left\{G_2(r)\left[j_0(\beta_* r) - \frac{9j_1(\beta_* r)}{\beta_* r} + \frac{32j_2(\beta_* r)}{(\beta_* r)^2}\right] + G_3(r)\left[j_0(\beta_* r) - \frac{j_1(\beta_* r)}{\beta_* r}\right]\right.\\
&\left.+4G_4(r)\left[\frac{j_1(\beta_* r)}{\beta_* r} - \frac{4j_2(\beta_* r)}{(\beta_* r)^2}\right]\right\}\Phi'(r)\mathrm{d}r - \beta_*\Gamma^{\Phi}.
\end{aligned} \tag{A.1.9}$$

Here, the functions $G_i(r)$ are

$$G_1(r) = \frac{1}{(\beta_0 r)^2}\left\{\left[\mathrm{i}\beta_0 r - 1 + (\beta_0 r)^2\right]\mathrm{e}^{\mathrm{i}\beta_0 r} - (i\alpha_0 r - 1)\mathrm{e}^{\mathrm{i}\alpha_0 r}\right\}, \tag{A.1.10}$$

$$G_2(r) = \frac{1}{(\beta_0 r)^2}\left\{\left[3(i\alpha_0 r - 1) + (\alpha_0 r)^2\right]\mathrm{e}^{\mathrm{i}\alpha_0 r} - \left[3(\mathrm{i}\beta_0 r - 1) + (\beta_0 r)^2\right]\mathrm{e}^{\mathrm{i}\beta_0 r}\right\}, \tag{A.1.11}$$

$$G_3(r) = \frac{1}{(\beta_0 r)^2}\left\{\left[3(1 - \mathrm{i}\beta_0 r) - 2(\mathrm{i}\beta_0 r)^2 + \mathrm{i}(\beta_0 r)^3\right]\mathrm{e}^{\mathrm{i}\beta_0 r} - \left[3(1 - i\alpha_0 r) - (\alpha_0 r)^2\right]\mathrm{e}^{\mathrm{i}\alpha_0 r}\right\}, \tag{A.1.12}$$

$$G_4(r) = \frac{1}{(\beta_0 r)^2}\left\{\left[9(1 - i\alpha_0 r) - 4(\alpha_0 r)^2 + \mathrm{i}(\alpha_0 r)^3\right]\mathrm{e}^{\mathrm{i}\alpha_0 r} - \left[9(1 - \mathrm{i}\beta_0 r) - 4(\beta_0 r)^2 + \mathrm{i}(\beta_0 r)^3\right]\mathrm{e}^{\mathrm{i}\beta_0 r}\right\}. \tag{A1.13}$$

A.3. The total scattering cross-section

The wave field outside an isolated inclusion consists of two parts: the incident field and the field scattered on the inclusion. Let us consider the diffraction of the plane shear wave propagating in the original matrix ($\beta_* = \beta_0$) on an isolated spherical inclusion. The field $u_i^s(x)$ scattered on the inclusion is the integral term in Eq. (3.1), and thus $u_i^s(x)$ has the form

$$u_i^s(x) = \int_v \left[\partial_j G_{ik}(x - x')C^1_{kjmn}\varepsilon_{mn}(x') + \rho_1\omega^2 G_{ik}(x - x')u_k(x')\right]\mathrm{d}x'. \tag{A.1.14}$$

Because integration here spreads over the region v occupied by the inclusion only, Eq. (A.1.14) defines the scattered field via the fields u_i and ε_{ij} inside the inclusion.

Let us consider the long-distant asymptotic of the scattered field. Using a standard technique of evaluation of the integral in Eq. (A.1.14) (Bohren and Huffman, 1983) we obtain that for large $|x|$ the following equation holds

$$u_i^s(x) \approx A_i(\hat{\mathbf{n}})\frac{\mathrm{e}^{\mathrm{i}\alpha_0 r}}{r} + B_i(\hat{\mathbf{n}})\frac{\mathrm{e}^{\mathrm{i}\beta_0 r}}{r}, \quad \hat{\mathbf{n}} = \frac{\mathbf{x}}{|x|}, \quad r = |x|. \tag{A.1.15}$$

Here, $A_i(\hat{\mathbf{n}})$ and $B_i(\hat{\mathbf{n}})$ are the vector amplitudes of the longitudinal and shear waves, scattered in the direction $\hat{\mathbf{n}}$. These amplitudes are expressed via the displacement and strain fields inside the inclusion by the following equations

$$A_i(\hat{\mathbf{n}}) = \hat{n}_i\hat{n}_k f_k(\alpha_0\hat{n}), \quad B_i(\hat{\mathbf{n}}) = (\delta_{ik} - \hat{n}_i\hat{n}_k)f_k(\beta_0\hat{n}), \tag{A.1.16}$$

$$f_k(q\hat{n}) = \frac{q^2}{4\pi\rho_0\omega^2}\left[\rho_1\omega^2\int_V u_k(x')\exp(-\mathrm{i}q\hat{\mathbf{n}}\cdot\mathbf{x}')\,\mathrm{d}x' + iqn_l C^1_{lkmn}\int_V \varepsilon_{mn}(x')\exp(-\mathrm{i}q\hat{\mathbf{n}}\cdot\mathbf{x}')\,\mathrm{d}x'\right],$$
$$(q = \alpha_0, \beta_0).$$

The normalized total scattering cross-section $Q_T(\omega)$ of the inclusion of a unit radius in the case of transversal wave propagation is defined by the equation

$$Q_T(\omega) = \frac{4}{\beta_0}\mathrm{Im}[\mathbf{m}\cdot\mathbf{B}(\mathbf{n})], \quad (\mathbf{m}\cdot\mathbf{n} = 0). \tag{A.1.17}$$

Here, $\mathbf{n}$ is the wave normal, $\mathbf{m}$ is the direction of the polarization vector.

Thus, the scattering cross-section is expressed via the forwarded scattering amplitude $\mathbf{B}(\mathbf{n})$ (the analogue of the "optical theorem" in electromagnetics (see Bohren and Huffman, 1983). Eqs. (A.1.16) and (A.1.17) together with Eqs. (4.12), (4.13) and (4.17) give

$$Q_T = \frac{4}{3}\beta_0\mathrm{Im}\left(\frac{\rho_1}{\rho_0}h_0 - \frac{\mu_1}{\mu_0}H_0\right), \tag{A.1.18}$$

where h_0 and H_0 coincide with h and H in Eqs. (4.13) and (4.17) when $\beta_* = \beta_0$.

It is possible to show that the short-wave limit of $Q_T(\omega)$ when $\omega \to \infty$ takes the form (paradox of extinction)

$$\lim_{\omega\to\infty} Q_T(\omega) = 2. \tag{A.1.19}$$

This limit does not depend neither on the properties of the inclusion nor on the properties of the matrix.

Appendix B. Structural factor *f* in Eq. (5.15) for the attenuation in the long-wave region

Consider a spatially homogeneous random set of spherical inclusions of unit radii ($a = 1$) in 3D-space, and let $V(x)$ be the characteristic function of the region occupied by the inclusions. The covariance $S_2(x) = \langle V(y)V(y+x)\rangle$ of the random function $V(x)$ is connected with the correlation function $\Phi(|x|)$ in Eqs. (3.30) and (3.22) by the relation

$$S_2(x) = \frac{3p}{4\pi}S_0(|x|) + p^2(1 - \Phi(|x|)), \tag{B.1}$$

where $S_0(|x|)$ is the volume of intersection of two spheres of unit radii if $|x|$ is the distance between its centers. It is shown in Torquato (1999) that for any realizable random function $V(x)$ the following inequality holds

$$\int \left[S_2(x) - p^2\right] \mathrm{d}x \geqslant 0. \tag{B.2}$$

After substituting in this equation function $S_2(x)$ and taking into account that $\int S_0(|x|)\,\mathrm{d}x = 4\pi/3$ we obtain

$$\int \left[S_2(x) - p^2\right] \mathrm{d}x = \frac{4}{3}\pi p - p^2 \int \Phi(|x|)\,\mathrm{d}x = \frac{4}{3}\pi p\left(1 - 3p\int_0^\infty \Phi(r)r^2\,\mathrm{d}r\right) = \frac{4}{3}\pi p f. \tag{B.3}$$

This equation together with Eq. (B.2) show that the structural factor

$$f = 1 - 3p\int_0^\infty \Phi(\zeta)\zeta^2\,\mathrm{d}\zeta \tag{B.4}$$

is a non-negative number for any realizable homogenous distribution of inclusions in space.

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